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Autour des représentations distinguées : La conjecture d'injectivité généralisée et Modèles symplectiques pour les groupes unitaires

Distinguished representations : The generalized injectivity conjecture and Symplectic models for unitary groups

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Résumé

Cette thèse est une contribution à l'étude des représentations distinguées et comporte deux parties indépendantes. La première s'intéresse à la Conjecture d'injectivité généralisée formulée par Casselman et Shahidi en 1998. La seconde est un travail en commun avec Dipendra Prasad.

Soit G un groupe connexe quasi-déployé défini sur un corps non-Archimédien de caractéristique nulle. On suppose que l'on se donne un sous-groupe parabolique standard de décomposition de Levi $P = MU$ ainsi qu'une représentation irréductible tempérée τ de M. Soit ν un élement dans le dual de l'algèbre de Lie de la composante déployée de M ; on le choisit dans la chambre de Weyl positive. La représentation induite $I_P^G(\tau_\nu)$ est appelée module standard. Quand la représentation τ est générique (pour un caractère non-dégénéré de U), i.e a un modèle de Whittaker, le module standard $I_P^G(\tau_\nu)$ est également générique. De plus, par un résultat de Rodier tout module standard générique a un unique sous-quotient générique.

Casselman et Shahidi ont conjecturé que cet unique sous-quotient générique apparaissait nécessairement comme sous-représentation dans le module standard $I_P^G(\tau_\nu)$. Cette conjecture a été démontrée dans le cas des groupes classiques $SO(2n+1), Sp(2n)$, et $SO(2n)$ quand P est un sous-groupe parabolique maximal de G, par Hanzer en 2010.

Dans notre travail, nous formulons et étudions ce problème dans le contexte d'un groupe réductif quasi-déployé quelconque en nous appuyant principalement sur la forme du support cuspidal, σ_λ, de cet unique sous-quotient irreductible générique. La forme explicite du support cuspidal est étudiée en utilisant la correspondance entre points résiduels dominants de la fonction μ et diagrammes de Dynkin pondérés. A partir de cette correspondance, nous introduisons la notion de *segments résiduels* et associons à un tel segment résiduel, un *ensemble de sauts*, une notion qui s'inspire des blocs de Jordan tels qu'étudiés par Moeglin et Tadic dans leur « Construction de séries discrètes pour les groupes p-adiques classiques ». Essentiellement, ces notions nous permettent de réduire notre argumentation au cas des séries principales non ramifiées.

Une fois la représentation irréductible générique cuspidale σ fixée, l'on peut étudier l'ensemble $\Sigma_\sigma := \left\{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$. C'est un système de racines dans le sous-espace vectoriel $a_{M_1}^*$.

Nous utilisons et prouvons l'existence de plongements stratégiques pour le sous-quotient irréductible générique lorsqu'il est de carré intégrable ; puis nous utilisons des opérateurs d'entrelacement à noyau non-générique. Ces outils nous permettent de prouver la Conjecture pour tout groupe connexe quasi-déployé tel que les composantes irréductibles de Σ_σ sont de type A, B, C ou D.

Le large cadre dans lequel nous avons démontré ces résultats semble de bon augure pour démontrer la conjecture en toute généralité.

Dans la deuxième partie de cette thèse nous étudions les modèles symplectiques pour les groupes unitaires. Nous prouvons d'abord qu'il n'existe pas de représentation cuspidale du groupe quasi-déployé $\mathrm{U}_{2n}(F)$ qui soit distinguée par son sous-groupe $\mathrm{Sp}_{2n}(F)$ pour F un corps local non-Archimédien. Nous prouvons ensuite le théorème équivalent pour un corps global : il n'existe pas de représentation cuspidale de $\mathrm{U}_{2n}(\mathbb{A}_k)$ qui ait une période symplectique non nulle pour k un corps de nombres ou corps de fonctions. Nous donnons une classification complète du groupe unitaire quasi-déployé en quatre variables sur un corps local ou global qui ont une période symplectique non-nulle en utilisant la correspondance Théta. Finalement, nous proposons une conjecture pour la classification de toutes les représentations d'un groupe unitaire quasi-déployé distinguées par $\mathrm{Sp}_{2n}(F)$.

Mots clé : représentations des groupes réductifs, modèles de Whittaker, modèles symplectiques

Abstract

This thesis is a contribution to the study of distinguished representations and is made up of two independant parts. The first is concerned with the Generalized Injectivity Conjecture formulated by Casselman and Shahidi in their paper « On irreducibility of standard modules for generic representations » published in 1998. The second is a joint work with Dipendra Prasad.

Let G be a quasi-split connected reductive group over a non-Archimedean local field F of characteristic zero. We assume we are given a standard parabolic subgroup P with Levi decomposition $P = MU$ as well as an irreducible, tempered representation τ of M. Let now ν be an element in the dual of the real Lie algebra of the split component of M ; we take it in the positive Weyl chamber. The induced representation $I_P^G(\tau_\nu)$ is called a standard module. When the representation τ is generic (for a non-degenerate character of U), i.e. has a Whittaker model, the standard module $I_P^G(\tau_\nu)$ is also generic. Further, by a result of Rodier any generic induced module has a unique irreducible generic subquotient.

Casselman and Shahidi have conjectured that the unique irreducible generic subquotient of a standard module $I_P^G(\tau_\nu)$ is necessarily a subrepresentation. This conjecture known as the Generalized Injectivity Conjecture was proved for the classical groups $SO(2n+1), Sp(2n)$, and $SO(2n)$ for P a maximal parabolic subgroup, by Hanzer in 2010.

In our work, we formulate and study this problem in the context of any quasi-split reductive group while mostly relying on the form of the cuspidal support, σ_λ of this unique irreducible generic subquotient. Explicit forms of the cuspidal support are studied using the correspondence between dominant residual points of the μ function and Weighted Dynkin diagrams. We introduce the notion of *residual segments* and associate to such residual segment, *set of Jumps* inspired by the notion of Jordan blocks studied in Moeglin and Tadic « Construction of discrete series for classical p-adic groups ». These notions somehow help in reducing the argumentation to the case of unramified principal series.

Once the irreducible generic cuspidal representation σ is fixed, one can study the set $\Sigma_\sigma := \left\{\alpha \in \Sigma_{\mathrm{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$. It is a root system of the subspace $a_{M_1}^*$.

We use and prove the existence of strategic embeddings for irreducible generic discrete series representations and further use intertwining operators with non-generic kernel. These tools allow us to prove the Generalized Injectivity Conjecture for any quasi-split connected reductive group such that the irreducible components of Σ_σ are of type A, B, C or D.

The larger framework in which we have studied this conjecture is a first step to prove it in full generality.

The second part of this thesis is concerned with symplectic models for unitary

groups. We prove that there are no cuspidal representations of the quasi-split unitary groups $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ for F a non-archimedean local field. We also prove the corresponding global theorem that there are no cuspidal representations of $\mathrm{U}_{2n}(\mathbb{A}_k)$ with nonzero period integral on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$ for k any number field or a function field. We completely classify representations of quasi-split unitary group in four variables over local and global fields with non-trivial symplectic periods using methods of theta correspondence. We propose a conjectural answer for the classification of all representations of a quasi-split unitary group distinguished by $\mathrm{Sp}_{2n}(F)$.

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Introduction

Introduction of the Thesis

0.0.1. Overview

The work in this thesis fits into the frame of harmonic analysis on connected reductive groups over local and global fields and more specifically in the study of particular classes of representations of these groups, namely distinguished representations.

Let F denote a local (non)-Archimedean (resp. global) field and **G** a connected reductive algebraic group defined over F. We will denote $G = \mathbf{G}(F)$ the corresponding group of F points.

For a subgroup H of a group G, a representation π of G is said to be distinguished by H if there exists a nonzero linear form $\ell : \pi \to \mathbb{C}$ such that $\ell(hv) = \ell(v)$ for all $h \in H$, and $v \in \pi$.

In some situations, we consider a slight generalization of the notion of distinction :
If χ is a quasicharacter of H, we define $\mathrm{Hom}_H(\pi, \chi) = \{\ell \in V^* | \ell(\pi(h)) = \chi(h)\ell, \forall h \in H\}$;
If $\mathrm{Hom}_H(\pi, \chi)$ is nonzero, we say that π is (H, χ)-distinguished.

The generic representations, leading the questions studied in the first chapter of this thesis are an example of (U, ψ)-distinguished representations, where ψ is a non-degenerate character of U the unipotent radical of a minimal parabolic P_0 of G.

After recalling some generalities and motivating the problem studied in the first chapter, we provide a statement of some of the main results in this manuscript. Some references on these notions are Borel « Lie Groups and Linear Algebraic Groups I » and Casselman's notes « Introduction to the theory of admissible representations of p-adic reductive groups ».

One of the main tools to exhibit and classify representations of reductive groups is parabolic induction.

Parabolic Induction

Recall that a parabolic subgroup **P** of **G** is a (Zariski) closed subgroup of **G** such that the quotient variety **G**/**P** is projective. We will denote P the F-points of a parabolic **P** defined over F. A realization of P as the product $P = MN$ with N the unipotent radical of P (and the quotient P/N is reductive) is refered as Levi decomposition of P, then M is a Levi subgroup of P.

Let A_0 be a maximal F-split torus of G and Δ a basis for the root system $\Sigma = \Sigma(A_0, G)$. The positive roots Σ^+ uniquely determines a minimal parabolic subgroup P_0 of G.

The standard Levi subgroup $M_0 = C_G(A_0)$ of P_0 is the centralizer in G of the torus A_0. As explained in Appendix A, the parabolic subgroups containing P_0 are called Δ-standard parabolic (often Δ is omitted when understood by context); such standard parabolic subgroups are (canonically) associated to subsets $\Theta \subset \Delta$. The parabolic subgroup corresponding to Θ is denoted P_Θ.

Let $P = MN$ be a parabolic subgroup of G, (σ, V) a smooth representation of M, and $X(\mathbf{M})_F$ the group of F-rational characters of M. The representation σ may be considered as a representation of P by extending σ trivially on N. The induced representation $Ind_P^G(\sigma_\nu)$ consists of the V-valued functions on G satisfying

$$f(mng) = \sigma(m) q^{\langle \nu + \rho_{\mathbf{P}}, H_M(m) \rangle} f(g)$$

where

$$H_M : M \to a_M = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$$

is the homomorphism defined by

$$q^{\langle \nu, H_M(m) \rangle} = |\chi_\nu(m)|_F$$

and $\rho_{\mathbf{P}}$ the half-sum of positive roots in $\mathbf{N}$ for all $\chi_\nu \in X(\mathbf{M})_F$ and all $m \in M$. The action is that of the regular right action, i.e, $Ind_P^G(\sigma_\nu)(g) f(g') = f(g'g)$ for $g, g' \in G$

A particular instance of induced representation where σ is an irreducible tempered representation of a standard Levi subgroup M of G, and $\nu \in (a_M^*)^+$ (a positive Weyl chamber) is called *a standard module*.

Langlands' classification and reducibilities of standard modules

There is a filtration of admissible (smooth and such that the space of K-fixed vectors is finite dimensional, for every compact open subgroup K) representations according to the growth properties of matrix coefficients :

cuspidal $\cap$ unitary $\subseteq$ discrete series $\subseteq$ tempered $\subseteq$ admissible

The matrix coefficients of an irreducible square-integrable (resp. tempered) representations are in $L^2(G \backslash Z)$ (resp. $L^{2+\epsilon}(G \backslash Z)$, for every $\epsilon > 0$), whereas the matrix coefficients of a cuspidal representation are compactly supported modulo the center.

One fundamental result arising in this classification is Langlands' Theorem stating that any irreducible admissible representation of a reductive group G is a Langlands quotient $J(P, \tau, \nu)$ (the unique irreducible quotient) of a standard module $I_P^G(\tau_\nu)$.

The reducibility of standard modules has been the subject of various works. First, it was conjectured (this is known as the *Standard Module Conjecture*) that when τ was generic, the standard module would be irreducible if and only if $J(P,\tau,\nu)$ was generic, then $J(P,\tau,\nu) \cong I_P^G(\tau_\nu)$. This was proven by Vogan in the context of real reductive groups and by Heiermann-Muic in HEIERMANN et MUIC 2006 for p-adic groups.

For classical groups, the reducibilities of standard modules were extensively studied, as reminded in Tadic's notes [see « Reducibility and discrete series, in the case of classical p-adic groups ; an approach based on examples », page 27].

Let S_n denote the symplectic or odd-orthogonal group of rank n. Let ρ and σ be unitarizable cuspidal representations of $GL_p(F)$ and S_q. Then, one denotes $\nu^\alpha\rho \rtimes \sigma$ the parabolically induced representation of S_{p+q}. Two fundamental results are :

1. If ρ is selfdual, there is exactly one $\alpha \geq 0$ for which $\nu^\alpha\rho \rtimes \sigma$ reduces (SILBERGER 1980b). This point will be denoted by $\alpha_{\rho,\sigma}$.
2. If ρ is selfdual and σ generic, then $\alpha_{\rho,\sigma} \in \{0, 1/2, 1\}$ (SHAHIDI 1990, SHAHIDI 1992)
3. Shahidi has proved that
$$\alpha_{\rho,\sigma} - \alpha_{\rho,\mathbf{1}} \in \mathbb{Z}$$
4. In general, it was conjectured (see Section 12 in MOEGLIN et TADIC 2002) that for σ generic, we have
$$\alpha_{\rho,\sigma} - \alpha_{\rho,\mathbf{1}} \in \mathbb{Z}$$
   A consequence of the last point is the *half-integer conjecture* cited in MOEGLIN 2014, which now follows from ARTHUR 2013 : Let $a \in \mathbb{R}$ be a non-negative real number such that $\nu^a\rho \rtimes \sigma$ reduces, then $a \in \frac{1}{2}\mathbb{Z}$.

Of a particular interest are the *generic* reducibilities. Let us be more precise on this notion of genericity.

Genericity

Let $\mathbf{A}_0$ be a fixed maximal split torus of $\mathbf{G}$, and P_0 be a fixed minimal parabolic subgroup of G having $\mathbf{A}_0$ as its split component. Let U be the unipotent radical of $\mathbf{P}_0$.

When $\mathbf{G}$ is a split classical group, the minimal parabolic $\mathbf{P}_0$ can be replaced by the Borel subgroup $\mathbf{B} = \mathbf{TU}$, with $\mathbf{T}$ the split torus consisting of diagonal matrices.

Once fixed $\mathbf{A}_0$, one defines a set of roots (resp. reduced roots) Σ (resp. Σ_{red}) , and a set of positive roots Σ^+ (resp. Σ^+_{red}) which depends on the choice of $\mathbf{P}_0$, and denote Δ the set of simple roots.

Roots of **G** with respect to $\mathbf{A}_0$ are non-trivial rational character α of $\mathbf{A}_0$ such that the eigenspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | Ad(a)X = \alpha(a)X \text{ for all a } \in A_0\}$$

in the Lie algebra $\mathfrak{g}$ of **G**, is non-trivial.

For each α in Σ, let $\mathbf{N}_\alpha$ be the subgroup of U whose Lie algebra is $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}(\mathfrak{g}_{2\alpha}$ may be trivial as in the case for split groups). Let us write $\mathbf{U} = \prod_{\alpha \in \Sigma^+_{\text{red}}} \mathbf{N}_\alpha$ with normal subgroup $\prod_{\alpha \in \Sigma^+ - \Delta} N_\alpha$ and isomorphism :

$$\mathbf{U} / \prod_{\alpha \in \Sigma^+ - \Delta} \mathbf{N}_\alpha \cong \prod_{\alpha \in \Delta} N_\alpha / \mathbf{N}_{2\alpha} \tag{0.1}$$

If $\psi_\alpha : N_\alpha / N_{2\alpha} \to \mathbb{C}$ is a smooth character for each $\alpha \in \Delta$, then $\psi = \prod_{\alpha \in \Delta} \psi_\alpha$ determines a character of U via the projection map $\mathbf{U} \to \mathbf{U} / \prod_{\alpha \in \Sigma^+ - \Delta} \mathbf{N}_\alpha$ and isomorphism (0.1) [1].

If each ψ_α is non-trivial, a character $\psi : U \to \mathbb{C}$ is called non-degenerate, or generic.

An irreducible admissible representation π of G is called generic (ψ-generic) if there exists a generic character ψ of U such that $\text{Hom}_G(\pi, \text{Ind}_U^G(\psi)) \neq 0$. This is equivalent to claim there exists a Whittaker functional $\lambda : V \to \mathbb{C}$ such that $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for all $u \in U$.

Let $\mathbb{C}_\psi$ denote the representation of U on $\mathbb{C}$ given by ψ. There is a map of G-spaces $W : V \to Ind_U^G(\mathbb{C}_\psi)$, $v \to W_v$ with $W_v(g) = \lambda(\pi(g)v)$. If π is irreducible, this map is an injection and is called a Whittaker model of π.

The Whittaker model of an irreducible representation was first introduced by Jacquet-Langlands as a natural local counterpart of the Fourier coefficients of automorphic forms on GL_2.

0.0.1.1. The origin : Fourier coefficients of automorphic forms

Let us assume F is global, $\mathbb{A}_F$ its ring of adeles. To obtain the Fourier expansion of an automorphic form ϕ on a group $G = \mathbf{G}(\mathbb{A}_F)$ one can choose a unipotent subgroup $N \subset G$ and try to write ϕ as a sum of terms $\sum_{\psi_N} F_{\psi_N}$ where the sum is over unitary characters ψ_N on $N(\mathbb{A}_\mathbb{F})$ trivial on $N(F)$ and each « Fourier coefficient » F_{ψ_N} is manifestly invariant with respect to the discrete subgroup $N(F) \subset N(\mathbb{A}_F)$. In practice, one would like to diagonalize the action of N. This works well if N is abelian but whenever N is non-abelian the expansion gets considerably more complicated.

1. Alternatively let $U/[U,U] \cong \prod_{\alpha \in \Delta} \mathbf{U}_\alpha(F)$ where each $U_\alpha = \mathbf{U}_\alpha(F)$ is the one dimensional subgroup generated by $exp(tX_\alpha)$, $t \in F$, X_α root vector. A character ψ of U is thus $\psi = \prod_\alpha \psi_\alpha$ for ψ_α character of U_α. Having fixed X_α, $\alpha \in \Delta$, each ψ_α is a character of F.

The next simplest type of Fourier coefficient F_{ψ_N} occurs when **N** is taken to be the maximal unipotent radical **U** in the Levi decomposition $\mathbf{B} = \mathbf{A}_0\mathbf{U}$ of the standard Borel subgroup **B** of **G**. It is called Whittaker coefficient, usually denoted by W, and Langlands proved that it factorises $W = \otimes_v W_v$ into a product of local Whittaker coefficients W_v for each local representation π_v. However, any global Fourier coefficient of an automorphic form does not exhibit a similar Euler product factorization.

To put it in a nutshell, Whittaker models are a non-abelian analogue of the Fourier transform.

Let us mention some major results and applications regarding genericity.

First, for an irreducible admissible representation of G, Shalika proved that the space of Whittaker functionals has dimension at most one. As a consequence, in the context of automorphic forms this uniqueness of local Whittaker models is specifically useful in computations of global integrals : when the integral can be expressed in terms of the global Fourier-Whittaker expansion of an automorphic form, then the local uniqueness expresses this integral as a product of local integrals, giving an Euler product.

A major application of Fourier coefficients of automorphic forms lies in Langlands' theory of automorphic L-functions and the transfer between automorphic representations of a group $\mathbf{G}(\mathbb{A})$ to another $\mathbf{G}'(\mathbb{A})$.

Local coefficients

This notion was introduced by Shahidi. Let σ be an irreducible admissible ψ_M-generic representation of M. We assume $P = MN = P_\theta$, and $N \subset U$ such that the character ψ_M is defined by means of restriction from a character ψ of U, assumed to be generic. We shall also assume (for the sake of simplicity) that all the ψ_α appearing in $\psi, \alpha \in \Delta$, are equal (as characters on F, see the footnote on the previous page). Let W be the Weyl group of $\mathbf{A}_0$ in **G**, i.e. the quotient of its normalizer by its centralizer.

Let $\tilde{w_0} \in W$ be such that $\tilde{w_0}(\theta) \subset \Delta$, let $N' = N_{\tilde{w_0}(\theta)} \subset U$, λ_M be a ψ_M-Whittaker functional for the space $\mathcal{H}(\sigma)$ of σ. For each f in the space $V(\sigma,\nu)$ of the representation $I = I_P^G(\sigma_\nu)$ we define : $\lambda_\psi(\nu,\sigma) = \int_{N'} \overline{\psi(n')}\lambda_M(f(w_0^{-1}n'))dn'$ where w_0 is a representative of $\tilde{w_0}$.

Clearly

$$\lambda(I(u)f) = \psi(u)\lambda(f)$$

where $\lambda_\psi(\nu,\sigma) = \lambda$; i.e λ is a ψ-generic Whittaker functional for I. One sees that $\lambda \neq 0$, moreover $\lambda_\psi(\nu,\sigma)$ is holomorphic for all ν (see Casselman et Shahidi 1998)

Now, there exists an intertwining operator $A(\nu,\sigma,w)$ (we won't give further details of this fundamental tool, see « Introduction to the theory of admissible representations of p-adic reductive groups » and Section 1.2 in the first chapter of this manuscript) between $I_P^G(\sigma_\nu)$ and $I^G_{P_{\tilde{w}(\theta)}}((w\sigma)_{w\nu})$. Let us denote $\lambda_\psi(w\nu, w\sigma)$ the Whittaker functional defined as above for $I^G_{P_{\tilde{w}(\theta)}}((w\sigma)_{w\nu})$, then $\lambda_\psi(w\nu, w\sigma)A(\nu,\sigma,w)$ is another non-zero Whittaker functional for $I_P^G(\sigma_\nu)$.

There is a theorem of Rodier RODIER 1972 which states that such functionals for I are unique up to scalars. Thus there exists a scalar $C_\psi(\nu,\sigma,w)$ called the local coefficient (see SHAHIDI 1981) attached to ν,σ and w

$$\lambda_\psi(\nu,\sigma) = C_\psi(\nu,\sigma,w)\lambda_\psi(w\nu,w\sigma)A(\nu,\sigma,w)$$

The uniqueness, up to scalars, of Whittaker functionals allows to determine such local coefficients. Their most important property is to appear in the *crude* functional equation (see SHAHIDI 1981).

In SHAHIDI 1990, Shahidi attached to each irreducible component r_i of the adjoint action of the L-group ${}^LM$ of M on Lie(${}^LU$) an L function $L(s,\pi,r_i)$ (along with ϵ and γ factors).

Let us assume F is global, $\mathbb{A}_F$ its ring of adeles, $G = G(\mathbb{A}_F)$, $P = MN$ maximal, and let ψ_M be a generic character of $U \cap M(F)\backslash U\cap M(\mathbb{A}_F)$. Let ϕ be a globally ψ_M -generic cusp form on M belonging to the space of the irreducible cuspidal representation $\pi = \otimes_v \pi_v$ of M. Write $\psi_M = \otimes_v\psi_{M,v}$; in this context (as explained in the Section 1.2.2) one can write the parameter $\nu \in (a_M^*)^+$ as $s\tilde{\alpha}$.

Theorem 1 (in SHAHIDI 1981)**.** *Let S be a finite set of places such that π_v and ψ_v are both unramified for every $v \notin S$. Then*

$$\prod_{i=1}^m L_S(is,\pi,r_i) = \prod_{v\in S} C_{\psi_v}(s\tilde{\alpha},\tilde{\pi}_v,w_0)\prod_{i=1}^m L_S(1-is,\pi,\tilde{r}_i)$$

Here $\psi = \otimes_v\psi_v$ is any generic character of $U(F)\backslash U(\mathbb{A}_F)$ which restrict to ψ_M and L_S is as usual the product of corresponding local L-functions at all $v \notin S$.

L-functions

From the above considerations on local coefficients, it becomes clear that the reducibility of standard modules is related to L-functions, let us mention the following proposition from Casselman-Shahidi's paper CASSELMAN et SHAHIDI 1998 :

Proposition 2 (Proposition 5.3 in CASSELMAN et SHAHIDI 1998)**.** *Let F be any local field. In the case of p-adic F, assume G satisfies the L-tempered Conjecture (see Conjecture 7.1 in* SHAHIDI *1990, proved in* HEIERMANN *et* OPDAM *2009). Let $P = MN$ be a maximal parabolic subgroup of G and fix a generic irreducible tempered representation τ of M. Suppose $Re(s) > 0$. A consequence of the Standard Module Conjecture is that $I_P^G(\tau_{s\tilde{\alpha}})$ is irreducible if and only if $\prod_{i=1}^m L(1-is,\tau,r_i)^{-1} \neq 0$. When F is Archimedean, the L-functions are those of Artin attached by Langlands (Theorem 6.1 of* VOGAN *1978)*

Having now given some motivations to study generic representations, let us introduce the problem considered in the first Chapter of this thesis : Vogan et al. soon noticed that most generic irreducible pieces appearing in the Jordan-Hölder composition series of standard module were subrepresentations.

One would therefore say that a standard module satisfies injectivity if all its irreducible subrepresentations are generic [Definition 3.1 in Casselman et Shahidi 1998].

Casselman and Shahidi first proposed the Injectivity Conjecture until Tadic suggested counter-examples for e.g for the groups GSp_8 and $SO(7)$, where a certain standard module has two (non-isomorphic) irreducible subrepresentations, only one of which is generic (see Section 3 in Casselman et Shahidi 1998). This is the genesis of the Generalized Injectivity Conjecture.

0.0.2. Statement of the main results

0.0.2.1. Main results for Chapter one

Let us recall that a quasi-split group over a field is a reductive group whose Borel subgroup is defined over the field F. A split group is a quasi-split group which has a maximal split torus defined over F. Examples of such groups are the classical groups (orthogonal, symplectic, and unitary groups).

Theorem 3 (Generalized Injectivity conjecture for quasi-split group). *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero) such that its root system is of type A, B, C or D (or product of these). Let π_0 be the unique irreducible generic subquotient of the standard module $I_P^G(\tau_\nu)$, then π_0 embeds as* a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

In fact, we prove this result for a larger class of groups but to quote it necessitates to introduce technical notations and we therefore refer the reader directly to the Introduction of Chapter one (Section 1.1).

This result was already known for classical groups by Hanzer Hanzer 2010. One of the aim of the present author's doctoral research has therefore been to find the appropriate reformulations and tools to reach our conclusions for any quasi-split group. Although our approach does not yet achieve the desired conclusion for all quasi-split groups, it is more likely to generalize.

An essential ingredient, which was not used by Hanzer in Hanzer 2010, in our method is an embedding result for irreducible generic discrete series of Heiermann-Opdam (Proposition 12).

We present a key embedding result for the unique irreducible generic discrete series subquotient of the generic standard module (see Proposition 56) relying on two extended Moeglin's Lemmas (see Lemmas 54 and 55) and this result of Heiermann-Opdam.

Capturing the essence of the proof requires a combinatorial analysis on the cuspidal support, σ_λ, of this unique irreducible generic subquotient. Explicit expressions of the parameter λ are studied using the correspondence between dominant residual points of the μ function and Weighted Dynkin diagrams. We introduce the notion of *residual segments* and associate to such residual segment, *set of Jumps* inspired by the notion of Jordan blocks studied in Moeglin and Tadic « Construction of discrete series for classical p-adic groups ». These notions somehow help in reducing the argumentation to the case of unramified principal series (i.e when the irreducible cuspidal representation σ is the trivial representation).

0.0.2.2. Main results for Chapter two

Let us come back to the notion of distinguished representations. Let $\mathbb{A}$ be the adele ring of a number field k. *Globally* the notion of distinction takes a slightly different flavour, a non-zero H-invariant linear form is a *period integral*.

$$\int_{H(k)\backslash H(\mathbb{A})} f(h)dh \not\equiv 0$$

for $f \in \Pi$, an automorphic representation of $G(\mathbb{A})$, for G a reductive algebraic group over the number field k, and H an algebraic subgroup of G defined over k.

For classical groups, analogously to Whittaker models, one can define symplectic models. In a joint work with Dipendra Prasad which consituted the second chapter of this thesis we have studied symplectic models for unitary groups.

One of the early results on symplectic periods is due to Heumos and Rallis who proved that there are no cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A})$ with nonzero symplectic period since in fact there are no generic representations of $\mathrm{GL}_{2n}(k_v)$ which are distinguished by $\mathrm{Sp}_{2n}(k_v)$. It was natural to wonder about the existence of symplectic period for quasi-split unitary groups since they also contain the symplectic group as a proper subgroup.

This work was an fruitful exercise to understand the interplay between local and global results. Let k be a number field, and K/k a quadratic extension. Our first main result was local :

Theorem 4. *Any representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ is a sub-quotient of a principal series representation of $\mathrm{U}(n,n)(F)$ induced from the Siegel parabolic (with Levi $\mathrm{GL}_n(E)$). In particular, a representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ cannot be cuspidal.*

And we achieve the same conclusion in a global setting :

Theorem 5. *Let Π be a cuspidal automorphic representation of $\mathrm{U}(W_n \otimes K)$. Then the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of* *the symplectic subgroup* $\mathrm{Sp}(W_n)$*, as well as on the symplectic subgroup* $\mathrm{Sp}(W_n)$ *is identically zero.*

A second part of this work is a playful application of *Theta correspondence* and the *theory of towers*. We obtain a complete classification of representations of the quasi-split unitary groups in four variables with non-trivial symplectic periods. Finally, an interpretation via Langlands parameters allow us to formulate a conjecture whose consequence would be that there are no tempered representations of $U(n,n)(F)$ distinguished by the symplectic groups $Sp_{2n}(F)$.

1. The Generalized Injectivity Conjecture

1.1. Introduction

1.1.1.

Let G ne a quasi-split connected reductive group over a non-Archimedean local field F of characteristic zero. We assume we are given a standard parabolic subgroup P with Levi decomposition $P = MU$ as well as an irreducible, tempered, generic representation τ of M. Let now ν be an element in the dual of the real Lie algebra of the split component of M ; we take it in the positive Weyl chamber. The induced representation $I_P^G(\tau,\nu) := I_P^G(\tau_\nu)$, called the standard module, has a unique irreducible quotient, $J(\tau_\nu)$, often named the Langlands quotient. Since the representation τ is generic (for a non-degenerate character of U, see the Section 1.2), i.e. has a Whittaker model, the standard module $I_P^G(\tau_\nu)$ is also generic. Further, by a result of RODIER 1972 any generic induced module has a unique irreducible generic subquotient.

In their paper CASSELMAN et SHAHIDI 1998 conjectured that :

(A) $J(\tau_\nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.

(B) The unique irreducible generic subquotient of $I_P^G(\tau_\nu)$ is a subrepresentation.

These questions were originally formulated for real groups by Vogan VOGAN 1978. Conjecture (B), was resolved in CASSELMAN et SHAHIDI 1998 provided the inducing data be cuspidal. After various progress in specific cases, Conjecture (A), known as the Standard Module Conjecture, was settled for quasi-split p-adic group in HEIERMANN et MUIC 2006 assuming the Tempered L Function Conjecture proven a few years later in HEIERMANN et OPDAM 2009.

The second conjecture, known as the Generalized Injectivity Conjecture was proved for classical groups $SO(2n+1), Sp(2n)$, and $SO(2n)$ for P a maximal parabolic subgroup, by Hanzer in HANZER 2010.

In the present work we prove the Generalized Injectivity Conjecture (Conjecture (B)) for any quasi-split connected reductive group provided the irreducible components of a certain root system (denoted Σ_σ) are of type A, B, C or D and some additional conditions satisfied (for the moment only) fo group of type A, B, C or D (see Theorem 6 below for a precise statement).

Following the terminology of Borel-Wallach [4.10 in BOREL et WALLACH 1999], for a standard parabolic subgroup P, τ a tempered representation and $\eta \in (a_M^*)^+$, a positive Weyl chamber, (P, τ, η) is refered as Langlands data, and η is the Langlands parameter, see the Definition 17 in this manuscript.

We will study the unique irreducible generic subquotient of a standard module $I_P^G(\tau_\eta)$ and make first the following reductions :

— τ is discrete series representation of the standard Levi subgroup M

— P is a maximal parabolic subgroup.

Then, η is written $s\tilde{\alpha}$, see the Subsection 1.2.2 for a definition of the latter.

Then, our approach has two layers : First we realized the generic discrete series τ as a subrepresentation of an induced module $I_{P_1\cap M}^M(\sigma_\nu)$ for a unitary generic cuspidal representation of M_1 (using Proposition 2.5 of Heiermann et Opdam 2009), and the parameter ν is dominant (i.e in some positive closed Weyl chamber) in a sense later made precise ; Using induction in stages, we can therefore embed the standard module $I_P^G(\tau_{s\tilde{\alpha}})$ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Let us denote $\nu + s\tilde{\alpha} := \lambda$. The unique generic subquotient of the standard module is also the unique generic subquotient in $I_{P_1}^G(\sigma_\lambda)$. By a result of Heiermann-Opdam [Proposition 2.5 of Heiermann et Opdam 2009], this generic subquotient appears as a subrepresentation of yet another induced representation $I_{P'}^G(\sigma'_{\lambda'})$ characterized by a parameter λ' in the closure of some positive Weyl chamber.

In an ideal scenario, λ and λ' are dominant with respect to P_1 (resp. P'), i.e. λ and λ' are in the closed positive Weyl chamber, and we may then build a bijective operator between those two induced representations using the dominance property of the Langlands parameters.

In case the parameter λ is not in the closure of the positive Weyl chamber, two alternatives procedures are considered : first, another strategic embedding of the irreducible generic subquotient in the representation induced from $\sigma''_{\lambda''}$ (relying on extended Moeglin's Lemmas) when the parameter λ'' (which depends on the form of λ) has a very specific aspect (this is Proposition 56) ; or (resp. and) showing the intertwining operator between $I_{P'}^G(\sigma'_{\lambda'})$ (resp. $I_{P_1}^G(\sigma''_{\lambda''})$) and $I_{P_1}^G(\sigma_\lambda)$ has non-generic kernel.

The embedding result (Proposition 56) requires to delve in a careful analysis of the Weyl group orbit of residual point or more precisely the parameter λ to be able to identify the elements λ'' in this orbit such that there will exist intertwining operators with non- generic kernel from $I_{P_1}^G(\sigma''_{\lambda''})$ to $I_{P_1}^G(\sigma_\lambda)$.

1.1.2.

In order to study a larger framework than the one of classical groups studied in Hanzer 2010, we will use the notion of *residual points* of the μ function (the μ function is the main ingredient of the Plancherel density for p-adic groups (see the Definition 10 and Subsection 1.2.1).

Indeed, as briefly suggested in the previous point, the triple (P_1, σ, λ), introduced above, plays a pivotal role in all the arguments developed thereafter, and of particular importance, the parameter λ is related to the μ function in the following ways :

— When σ_λ is a residual point for the μ function (abusively one can say that λ is a residual point), the unique irreducible generic subquotient in the module induced from σ_λ is discrete series (A result of Heiermann in HEIERMANN 2004, see Proposition 11).

— Once the cuspidal representation σ is fixed, one can study the set Σ_σ, a root system of the subspace $a^*_{M_1}$ defined using the μ function. This is where stands the particularity of our method, to deal with all possible standard modules, we needed an explicit description of this parameter λ lying in $a^*_{M_1}$. Thanks to Bala-Carter theory, such descriptive approach is made possible. Indeed, we have bijective correspondences between the following sets explained in Section 1.5 :

$\{$dominant residual point$\}$

$\{$Weighted Dynkin diagram(s)$\}$

The notion of Weighted Dynkin diagram is established and recalled in the Appendix F.1.

We use this correspondence to express the dominant residual point in explicit terms by what we call *residual segments* generalizing the classical notion of segments (of Bernstein-Zelevinsky). We associate to such a residual segment *set(s) of Jumps* (a notion connected to that of Jordan blocks elements in the classical groups setting of Moeglin-Tadic in MOEGLIN et TADIC 2002).

Further, the μ function is intrinsically related to the *intertwining operators* mentioned in the previous subsection, see the point 1.1.6.

1.1.3.

Let us briefly comment on the organisation of this manuscript, therefore giving a general overview of our results and the scheme of proof. In subsequent points 1.1.5 and 1.1.6, we will give details on the ingredients of proofs.

In Section 1.3, we formulate the problem in an as broad as possible context (any quasi-split reductive p-adic group G) and prove a few results on intertwining operators.

As M.Hanzer in HANZER 2010, we distinguish two cases : the case of a generic discrete series subquotient, and the case of a non-discrete series generic subquotient. As stated in 1.1.2, the case of discrete series subquotient corresponds to σ_λ (in the cuspidal support of the generic discrete series) being a residual point.

As just stated in 1.1.2, our approach uses the bijection between Weyl group orbits of residual points and weighted Dynkin diagrams as studied in OPDAM 2004 and explained in the Appendix F.

Through this approach, we can explicit the Langlands parameters of subquotients of the representations $I_{P_1}^G(\sigma_\lambda)$ induced from the generic cuspidal support σ_λ and classify them using the order on parameters given in $a^*_{M_1}$ as in chapter XI, Lemma 2.13 in BOREL et WALLACH 1999. In particular, the minimal element for this order (in a sense later made precise) characterizes the unique irreducible generic non-discrete series subquotient, see Theorem 35.

Although requiring to get acquainted with the notions of residual points, and then residual segments, our methods have two advantages.

The first is proving the Generalized Injectivity Conjecture for all quasi-split reductive groups (provided a certain construction of the standard Levi subgroup M_1 and the irreducible components of a certain root system to be of type A, B, C or D ; we have verified those conditions when the root system of the quasi-split (hence reductive) group is of type A, B, C or D), and recovering the results of Hanzer through alternative proofs.

In particular, a key ingredient (which was not used by Hanzer in HANZER 2010) in our method is an embedding result of Heiermann-Opdam (Proposition 12).

The second is a self-contained and uniform (in the sense that cases of root systems of type B, C and D are all treated in the same proofs) treatment.

Although based on the ideas of Hanzer in HANZER 2010, our approach is more likely to generalize to all quasi-split groups.

In the Subsection 1.8.1 we present an embedding result for the unique irreducible generic discrete series subquotient of the generic standard module (see Proposition 56) relying on two extended Moeglin's Lemmas (see Lemmas 54 and 55) and the result of Heiermann-Opdam (see Proposition 12).

This embedding result is used in Section 1.8 to prove the Generalized Injectivity Conjecture for discrete series generic subquotient, first when P is a maximal parabolic subgroup and secondly for *any parabolic subgroup* in Section 1.8.4.

In Section 1.9, we continue with the case of non-discrete series subquotients, and further conclude with the case of the standard module induced from a tempered representation τ in Corollary 63 and Corollary 66.

1.1.4.

Let us come back on the structure of the proof.

Following Hanzer in HANZER 2010, we have considered successively the case of generic irreducible discrete series subquotients and generic irreducible non-discrete series subquotients (i.e. tempered, or non-tempered).

Fortunately, in the context of quasi-split reductive groups, two crucial results can be used : first, the irreducible generic discrete series subquotient embeds in $I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is a residual point (as proved by Heiermann in HEIERMANN 2004) and secondly, λ satisfies a certain positivity condition, i.e. is in the closure of the positive Weyl chamber as proven in Proposition 2.5 in HEIERMANN et OPDAM 2009 (see Proposition 12).

In fact, once a cuspidal representation σ of the Levi subgroup M_1 is fixed, we consider the Weyl group orbit of the residual point σ_λ : in this orbit there is a unique λ parameter which is dominant, i.e. in the closure of the positive Weyl chamber. This dominant parameter in the dual of the Lie algebra $a_{M_1}^*$ will be described using the bijection between weigthed Dynkin diagrams and dominant residual points. In the canonical basis of this vector space the parameter is written as a string of (half)-integers [1] which depends on the weights of the Dynkin diagram. Such string of (half)-integers will be called *residual segments*, where the notion of *segments* stands in analogy with the notion introduced by Bernstein-Zelevinsky in Bernstein et Zelevinsky 1977.

More precisely, let α be a root in the set of reduced roots of A_{M_1} in Lie(G) and $(M_1)_\alpha$ be the centralizer of $(A_{M_1})_\alpha$ (the identity component of the kernel of α in A_{M_1}), we will consider the set

$$\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

it is a subset of $a_{M_1}^*$ which is a root system in a subspace of $a_{M_1}^*$(cf Silberger 1981 3.5) and we suppose the irreducible components of Σ_σ are of type A, B, C or D.

1.1.5.

Having defined the root system Σ_σ, let us present the main result of this paper :

Theorem 6 (Generalized Injectivity conjecture for quasi-split group)**.** *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero) such that its root system is of type A, B, C or D (or product of these). Let π_0 be the unique irreducible generic subquotient of the standard module $I_P^G(\tau_\nu)$, then π_0 embeds as* a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

Theorem 7 (Generalized Injectivity conjecture for quasi-split group)**.** *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero). Let π_0 be the unique irreducible generic subquotient of the standard module $I_P^G(\tau_\nu)$, let σ be an irreducible, generic, cuspidal representation of M_1 such that a twist by an unramified real character of σ is in the cuspidal support of π_0.*

Suppose that all the irreducible components of Σ_σ are of type A, B, C or D, then, under certain conditions on the Weyl group of Σ_σ (explained in Section 1.8.1, in particular Corollary 47), π_0 embeds as a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

Theorem 6 results from 7. The Theorem 7 could be true when the root system of the group G contains components of type E, F and G (We have not yet proven these conditions are satisfied in this case).

The proof of Theorem 7 is done in several steps. First, we prove it for the case of an irreducible generic discrete series subquotient assuming τ discrete series, and Σ_σ irreducible in Proposition 57.

1. The half-integers are precisely those numbers that are half of an odd integer. The notation (half)-integers means either half-integers or integers

We use this latter result for the case of a tempered or non tempered irreducible generic subquotient in Proposition 61 ; and also for the case of standard modules induced from non-maximal standard parabolic (Theorems 60 and 62). Then, the case of τ tempered follows (Corollary 63).

The case of Σ_σ reducible is done in Section 1.20.

For each α in Σ_σ, the reducibility point Λ of $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_\Lambda)$ (see Proposition 24 and Example 2 following this Proposition) determines the type of weighted Dynkin diagram to be considered to evaluate the coordinates (in a basis of $(a_{M_1}^G)^*$, so that the elements of the root system Σ_σ are written in this basis as in Bourbaki *Groupes et Algèbres de Lie, Chapitre 4,5, et 6*) of the parameter λ corresponding to the residual point σ_λ.

The Proposition 24 also gives conditions on the rank of the root system Σ_σ, for σ_λ to be a residual point.

Remember we consider a standard module $I_P^G(\tau_{s\tilde{\alpha}})$ with $P = MU$. Therefore,

$$\Sigma_\sigma^M = \{\alpha \in \Sigma_{red}^M(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

will be the second main root system at the center of our analysis (see Section 1.5.3).

Typically, if Σ_σ is irreducible and $\mathcal{T}$ denotes its type, let $\Delta_\sigma := \{\alpha_1, \ldots, \alpha_d\}$ be the basis of Σ_σ (following our choice of basis for the root system of G).

Let us consider maximal standard Levi subgroup of G, $M = M_\Omega \supset M_1$, corresponding to subsets $\Omega = \Delta - \{\beta\} \subset \Delta$ where β is a non extremal simple root of the Dynkin diagram of G. The subset Ω is a union of two connected components, and Σ_σ^M is a direct sum of two irreducible components $\Sigma_{\sigma,1}^M \bigcup \Sigma_{\sigma,2}^M$ of type A and $\mathcal{T}$.

1.1.6.

With the context and restrictions of the last paragraph of 1.1.5, let us explain the ingredients of the proof.

Let τ be an irreducible generic discrete series representation of a standard maximal Levi subgroup M of G. Using the result of Heiermann-Opdam (Proposition 12), it can be embedded in $I_{P_1\cap M}^M(\sigma_\nu)$. The representation σ is unitary, the parameter ν is in $\overline{(a_{M_1}^{M\,*})^+}$ and σ_ν is a residual point for μ^M. The representation σ depends on the representation τ.

We will first assume Σ_σ is irreducible and prove the result under this restriction. The case of Σ_σ reducible is considered in Proposition 65.

Since σ_ν is in the cuspidal support of the generic discrete series τ, applying the condition on the rank mentioned in the last paragraph of 1.1.5 (see Proposition

24) we have : $rk(\Sigma_\sigma^M) = d_1 - 1 + d_2$ and write (as in the last paragraph of 1.1.5)

$$\Sigma_\sigma^M := A_{d_1-1} \bigcup \mathcal{T}_{d_2}$$

such that ν corresponds to *residual segments* ν_A and $\nu_{\mathcal{T}}$. The coordinates of these two vectors (of respective length d_1 and d_2) are computed using the weights of Weighted Dynkin diagrams (see our definition of residual segments in Definition 25).

Further, we twist the discrete series τ with

$$s\tilde{\alpha} \in {a_M^*}^+$$

this twist is added on the linear part (i.e corresponding to A_{d_1-1}). Consequently, $\nu_{\mathcal{T}}$ is left unchanged and is thus $\lambda_{\mathcal{T}}$, whereas ν_A becomes $\lambda_A = \nu_A + s\tilde{\alpha}$.

In this very specific context, we can characterize the set of two residual segments

$$(\nu_A, \nu_{\mathcal{T}})$$

Let us denote W_σ the Weyl group of Σ_σ. The first residual segment of type A_{d_1-1} is uniquely characterized by two (half)-integers a, b with $a > b$ and the residual segment of type $\mathcal{T}_{d_2}$ is uniquely characterized by a tuple $\underline{n}$.

We call each such triple $(a, b, \underline{n})$ a *cuspidal string* and call W_σ-cuspidal string the orbit of the Weyl group W_σ of this *cuspidal string* (see the definitions in Section 1.5.3).

An example of this construction consists in the representation of a standard Levi subgroup $GL_{k\times d_1} \times G(k')$ of a classical group $G(n)$ of rank n. It is a tensor product of a Steinberg representation (of $GL_{k\times d_1}$) with π an irreducible generic discrete series of a classical group of smaller rank, $G(k')$, $n = 2kd_1 + k'$, $d_1 = a - b + 1$:

$$St_{d_1}(\rho)|.|^{\frac{a+b}{2}} \otimes \pi$$

The irreducible generic discrete series π corresponds to a residual segment $(\underline{n})$.

If we obtain from the vector of coordinates of $(\lambda_A, \lambda_{\mathcal{T}})$ a residual segment of length $d = \text{rk}(\Sigma_\sigma)$ and type $\mathcal{T}$, σ_λ is a residual point for μ^G and the induced representation $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient (as explained in Proposition 24) ; this is the case where the unique irreducible generic subquotient is discrete series (by Theorem 33).

The Weyl group W_σ fixes the irreducible unitary cuspidal representation σ and acts on the parameter λ in $a_{M_1}^*$. In the W_σ-orbit $(a, b, \underline{n})$, we will find a *cuspidal string* $(a', b', \underline{n'})$ such that the unique irreducible generic subquotient (tempered or non-tempered), denoted $I_{P'}^G(\tau'_{\nu'})$ embeds in $I_{P_1}^G(\sigma_{(a',b')+(\underline{n'})}) := I_{P_1}^G(\sigma(a', b', \underline{n'}))$.

The parameter ν' corresponds to the minimal element for the order on parameters in $a_{M_1}^*$ given in Chapter XI, Lemma 2.13 in Borel et Wallach 1999, and this minimality condition is used in Appendix G to identify the form of the cuspidal string $(a', b', \underline{n'})$ in the W_σ orbit of the cuspidal string $(a, b, \underline{n})$.

This is where the μ-function intervenes a second time, since this function enters in the definition of *intertwining operators*. Intertwining operators with non-generic kernel (see Proposition 18) allow us to transfer generic irreducible pieces (such as $I_{P'}^G(\tau'_{\nu'})$) from $I_{P_1}^G(\sigma(a', b', \underline{n}'))$ to $I_{P_1}^G(\sigma((a, b, \underline{n}))$.

Since the latter induced module also contains $I_P^G(\tau_{s\tilde{\alpha}})$, by multiplicity one the irreducible generic subquotient, we conclude that $I_P^G(\tau_{s\tilde{\alpha}})$ contains $I_{P'}^G(\tau'_{\nu'})$ as a subrepresentation.

1.1.7.

Let us come back on the case of an irreducible discrete series generic subquotient.

It requires a more careful analysis of the properties of *residual segments*. As explained in Section 1.5, to a residual segment $(\underline{n})$, we associate a *set of Jumps* (a notion very similar to that of *Jordan blocks* from Moeglin-Tadic Moeglin et Tadic 2002) ; and then using extended Moeglin's Lemmas (see Lemmas 54 and 55), and the result of Heiermann-Opdam (Proposition 12), we prove an embedding result, Proposition 56 (equivalent to the Proposition 3.1 in Hanzer 2010 for classical groups) used to prove the generalized injectivity conjecture in this context.

1.1.8.

Finally, the reader will notice that our main results (Proposition 57, Theorems 60 and 62, Proposition 61) are formulated such that $s\tilde{\alpha}$ (resp. $\underline{s}$ in case M is not maximal) is in $\overline{a_M^{*+}}$ rather than in a_M^{*+}.

Let us recall the difference between the two. Let M be M_Θ, where $\Theta \subset \Delta$, is the set of simple roots in Lie(M). For $\underline{s}$ (resp. $s\tilde{\alpha}$) to be in a_M^{*+} means $\left\langle \underline{s}, \check{\beta} \right\rangle > 0$ for all roots β in $\Delta - \Theta$, whereas $\left\langle \underline{s}, \check{\beta} \right\rangle = 0$ for all roots in Δ^M (roots in Lie(M)) (resp. $s > 0$).

If the parameter $\underline{s}$ is in $\overline{a_M^{*+}}$, it means we may also have $\left\langle \underline{s}, \check{\beta} \right\rangle = 0$, for some linear combinations of simple roots in $\Delta - \Theta$ (resp. $s \geq 0$).

1.1.9.

The methods of proof developed thereafter will be illustrated under the following restriction : Let n be the rank of the group $G(n)$, and let assume the form of the Levi subgroup M_1 is isomorphic to $\prod_i \underbrace{GL_{k_i}}_{d_i \text{ times}} \times G(k_0)$ where the multisets $\{k_0; (k_1, \ldots, k_r)\}, n = k_0 + d_1 k_1 + \ldots d_r k_r, k_0 \geq 0$, index the conjugacy classes of Levi subgroups of the group $G(n)$. This condition is satisfied for all classical groups and their variants (we borrow this expression from Moeglin Mœglin 2011).

In this context, because of the restriction on the form of the Levi subgroup M_1, the generic representation σ_λ of M_1 which lies in the cuspidal support takes the form :

$$\rho|.|^{a} \otimes \rho|.|^{a-1} \ldots \rho|.|^{b} \otimes \underbrace{\sigma_2|.|^{\ell_2} \ldots \sigma_2|.|^{\ell_2}}_{n_{\ell_2} \text{ times}} \ldots \underbrace{\sigma_2|.|^{0} \ldots \otimes \sigma_2|.|^{0}}_{n_{0,2} \text{ times}} \otimes \ldots \otimes$$
$$\underbrace{\sigma_r|.|^{\ell_r} \ldots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \ldots \underbrace{\otimes \sigma_r|.|^{0} \ldots \otimes \sigma_r|.|^{0}}_{n_{0,r} \text{ times}} \otimes \sigma_c$$

where $\sigma_i \;\; i = 2, \ldots, r$ (resp. ρ) are unitary cuspidal representations of GL_{k_i} (resp. GL_{k_1}) and σ_c a cuspidal representation of $G(k_0)$.

The tuple $(a, \ldots, b)$ is a decreasing sequence of (half)-integers corresponding to a residual segment of type A; whereas for each $i \geq 2$, the residual segment (of type B, C or D) is $(\underline{n_i}) := (0, \ldots, 0, n_{\ell_i}, \ldots, n_{1,i}, n_{0,i})$.

Since we are dealing with a generic cuspidal support, the reducibility point (0,1/2, or 1) of the induced representation of $G(k_0 + k_i)$: $I_{P_1}^{G(k_0+k_i)}(\rho|.|^{s} \otimes \sigma_c)$ explicitely determine the form of the parameters, as explained in Proposition 24 and the Example 2 following this Proposition.

Therefore, a corollary of our Theorem 6 is the following :

Corollary 1. *The generalized injectivity conjecture is true for all classical groups and their variants.*

In Appendix D, we illustrate our method of proof on GL_n and further prove the Generalized Injectivity Conjecture for its derived subgroup SL_n. More generally, for $G \subset \tilde{G}$, having the same derived subgroup, it is enough to prove the Generalized Injectivity conjecture for $\tilde{G}$, then the result follows for G (Theorem 75). In particular, we will prove the Generalized Injectivity Conjecture for odd and even Spin groups, since we prove it for odd and even GSpin.

Further, in the Subsection D.3, the reader will find most of our results proved in the context of classical groups and their variants. Most of these results were already known by the work of Hanzer in Hanzer 2010, we recover them using similar tools but in a novel way; in particular we are relying on the result of Heiermann-Opdam (Proposition 12).

1.2. Preliminaries

Let F be a non-Archimedean local field of characteristic 0. Denote by G the group of F-rational points of a quasi-split connected reductive group defined over F.

Fix a minimal parabolic subgroup P_0 (which is a Borel B since G is quasi-split) with Levi decomposition $P_0 = M_0U_0$ and A_0 a maximal split torus (over F) of M_0. P is said to be standard if it contains P_0.
More generally, if P rather contains A_0, it is said to be semi-standard. Then P contains a unique Levi subgroup M containing A_0, and M is said to be semi-standard.

For a semi-standard Levi subgroup M, we denote $\mathcal{P}(M)$ the set of parabolic subgroups P with Levi factor M.

We denote by A_M the maximal split torus in the center of M, $W = W^G$ the Weyl group of G defined with respect to A_0 (i.e. $N_G(A_0)/Z_G(A_0)$). The choice of P_0 determines an order in W, and we denote by w_0^G the longest element in W.

If Σ denote the set of roots of G with respect to A_0, the choice of P_0 also determines the set of positive roots (resp., negative roots, simple roots) which we denote by Σ^+ (resp., Σ^-, Δ).

To a subset $\Theta \subset \Delta$ we associate a standard parabolic subgroup $P_\Theta = P$ (see the Appendix A) with Levi decomposition MU, and denote A_M the split component of M. We will write a_M^* for the dual of the real Lie-algebra a_M of A_M, $(a_M)^*_{\mathbb{C}}$ for its complexification and a_M^{*+} for the positive Weyl chamber in a_M^* defined with respect to P.
Further $\Sigma(A_M)$ denotes the set of roots of A_M in Lie(G). It is a subset of a_M^*. For any root $\alpha \in \Sigma(A_M)$, we can associate a coroot $\check{\alpha} \in a_M$. For $P \in \mathcal{P}(M)$, we denote $\Sigma(P)$ the subset of positive roots of A_M relative to P. We write Δ^M for the roots of Δ in M, and Δ_M the subset of $\Sigma(A_M)$ consisting in the non-trivial restrictions of elements in Δ.

Let Rat(M) be the group of F-rational characters of M, we have :

$$a_M^* = \text{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R} \text{ and } (a_M)^*_{\mathbb{C}} = a_M^* \otimes_{\mathbb{R}} \mathbb{C}$$

For $\chi \otimes r \in a_M^*$, $r \in \mathbb{R}$, , and λ in a_M, the pairing $a_M \times a_M^* \to \mathbb{R}$ is given by : $\langle \lambda, \chi \otimes r \rangle = \lambda(\chi).r$
Following Waldspurger 2003 we define a map

$$H_M : M \to a_M = \text{Hom}(\text{Rat}(M), \mathbb{R})$$

such that

$$|\chi(m)|_F = q^{-\langle \chi, H_M(m) \rangle}$$

for every F-rational character χ in a_M^* of M, q being the cardinality of the residue field of F.

We denote by $X(M)$ the group of unramified characters of M. This space consists of all continuous characters of M into $\mathbb{C}^*$ which are trivial on the distinguished subgroup $M^1 = \bigcap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}|\chi(.)|$ of M. Its relation with $(a_M)^*_{\mathbb{C}}$ is given by the surjection

$$(a_M)^*_{\mathbb{C}} \to X(M)$$

which associates the character $\chi_\nu = q^{-\langle \nu, H_M(.)\rangle}$ to the element ν in $(a_M)^*_{\mathbb{C}}$. The kernel of this map if of the form $\frac{2\pi i}{logq}\Lambda$, for a certain lattice Λ of $(a_M)^*$. This surjection gives $X(M)$ the structure of a complex algebraic variety, where $X(M) \cong (\mathbb{C}^*)^d, d = \dim_{\mathbb{R}} a_M$. Thus there are notions of polynomial and rational functions on $X(M)$.

Let us recall those notions as stated in [Waldspurger 2003, IV.1].

Set $\mathcal{O}_{\mathbb{C}} = \{\pi \otimes \chi, \chi \in X(M)\}$, where $\pi \otimes \chi$ is an isomorphism class of representations. Denote B the algebra of polynomials on the algebraic variety $X(M)$. A function $f : \mathcal{O}_{\mathbb{C}} \to \mathbb{C}$ is said to be polynomial if there exists $b \in B$ such that $f(\pi \otimes \chi) = b(\chi)$ for any χ in $X(M)$. If $\mathcal{U}$ is a open set of $\mathcal{O}_{\mathbb{C}}$, $f : \mathcal{U} \to \mathbb{C}$ is said to be rational if there exists $b_1, b_2 \in B$ such that $b_1(\chi)f(\pi \otimes \chi) = b_2(\chi)$ and $b_1(\chi)$ is non-zero for any $\chi \in X(M)$ such that $\pi \otimes \chi \in \mathcal{U}$. Then we analytically continue f to $\{\pi \otimes \chi, \chi \in X(M), b_1(\chi) \neq 0\}$.

For π a smooth representation of M and ν in $a^*_{M,\mathbb{C}}$ we define by π_ν the representation of M :

$$\pi_\nu(m) = \pi \otimes \chi_\nu(m) = q^{-\langle \nu, H_M(m)\rangle}\pi(m)$$

Genericity For ψ a non-degenerate character of U, an admissible representation of G, (π, V) is said to be ψ-generic if there exists a non-zero linear functional $\lambda : V \to \mathbb{C}$ such that $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for all $u \in U$. Such λ is called a Whittaker functional.

By Shahidi 2010 Sections 3.3 and 1.4, we can fix a non-degenerate character ψ of U which, for every Levi subgroup M, is compatible with $w_0^G w_0^M$. We will still denote ψ the restriction of ψ to $M \cap U$. Every generic representation π of M becomes generic with respect to ψ after changing the splitting in U. Throughout this paper, generic means ψ-generic. When the groups are quasi-split and connected, by a theorem of Rodier, the standard ψ-generic modules have exactly one ψ-generic irreducible subquotient.

Induced representations and standard intertwining operators Let us assume that (σ, V) is an admissible complex representation of M, consider the space of smooth functions on G which transform under left multiplication by P

according to $\delta^{1/2}\sigma$:

$$V(\sigma) = \left\{ f \in C^\infty(G,V) | f(mug) = \sigma(m)\delta_P(m)^{1/2} f(g) \quad \forall m \in M, u \in U, g \in G \right\}$$

The group G acts on this space by right translations, and the corresponding representation, unitarily induced from σ, is denoted $I_P^G(\sigma)$.

The set of equivalence classes of irreducible representations of G will be denoted by $\mathcal{E}(G)$, the subset of cuspidal representations by $\mathcal{E}_c(G)$.

Given π in $\mathcal{E}(G)$ there exist a semi-standard parabolic subgroup $P = LU$ and a cuspidal representation σ in $\mathcal{E}_c(L)$ such that π a subquotient of $I_P^G(\sigma)$. The G-conjugacy class of L and σ is uniquely determined by π. It is called the cuspidal support of π.

We adopt the convention that the isomorphism class of (σ, V) is denoted by σ. If χ_ν is in $X(G)$, then we write $(\sigma_\nu, V_{\chi_\nu})$ for the representation $\sigma \otimes \chi_\nu$ on the space V.

Let (σ, V) be an admissible representation of finite length of M, a Levi subgroup containing M_0 a minimal Levi subgroup, centralizer of the maximal split torus A_0. Let P and P' be in $\mathcal{P}(M)$. Consider the intertwining integral :

$$(J_{P'|P}(\sigma_\nu)f)(g) = \int_{U \cap U' \backslash U'} f(u'g)du' \quad f \in I_P^G(\sigma_\nu)$$

where U and U' denote the unipotent radical of P and P', respectively.

For ν in $X(M)$ with $\mathrm{Re}(\langle \nu, \check{\alpha} \rangle) > 0$ for all α in $\Sigma(P) \cap \Sigma(P')$ the defining integral of $J_{P'|P}(\sigma_\nu)$ converges absolutely. Moreover, $J_{P'|P}$ defined in this way on some open subset of $\mathcal{O} = \{\sigma_\nu | \nu \in X(M)\}$ becomes a rational function on $\mathcal{O}$ (Waldspurger 2003 Theorem IV 1.1). Outsides its poles, this defines an element of

$$\mathrm{Hom}_G(I_P^G(V_\chi), I_{P'}^G(V_\chi))$$

Moreover, for any χ in $X(M)$, there exist an element v in $I_P^G(V_\chi)$ such that $J_{P'|P}(\sigma_\chi)v$ is not zero (Waldspurger 2003, IV.1 (10))

In particular, for all ν in an open subset of a_M^*, and $\overline{P}$ the opposite parabolic subgroup to P, we have an intertwining operator

$$J_{\overline{P}|P}(\sigma_\nu) : I_P^G(\sigma_\nu) \to I_{\overline{P}}^G(\sigma_\nu)$$

and for ν in $(a_M^*)^+$ far away from the walls it is defined by the convergent integral :

$$(J_{\overline{P}|P}(\sigma_\nu)f)(g) = \int_{\overline{U}} f(ug)du$$

The intertwining operator is meromorphic in ν and the map $J_{\overline{P}|P} J_{P|\overline{P}}$ is a scalar. Its inverse equals the Harish-Chandra μ function up to a constant and will be denoted $\mu^G(\sigma_\nu)$:

$$J_{\overline{P}|P} J_{P|\overline{P}}(\sigma_\nu) = \frac{c}{\mu^G(\sigma_\nu)}$$

Some subsets of the Weyl group Let M_1 be a standard Levi subgroup of G.

Let us define $W_{\text{std}}(M_1)$ as the set of elements w in W of minimal length in their right classes modulo W^{M_1} and such that $w^{-1}M_1w$ is again a standard Levi. Let us, now, define elementary symmetries following the treatment in WALSPURGER.JL 1995, I.1.7.

In general, it is known that the Weyl group W^G is a Coxeter group generated by the symmetries relative to the simple roots.

Let us denote $\Sigma_{red}(A_{M_1}, G)$ the set of indivisible roots in $\Sigma(A_{M_1}, G)$.
Set $\ell(w) = \#\{\alpha \in \Sigma(A_{M_1}, G) : \alpha > 0, w\alpha < 0\}$. This defines a length function $\ell : W_{\text{std}}(M_1) \to \mathbb{N}$.

We define $(M_1)_\alpha$ to be the centralizer of $(A_{M_1})_\alpha$ (the identity component of the kernel of α in A_{M_1}). If α is in Δ_{M_1} (generating set for $a^*_{M_1}$; following the notations of WALSPURGER.JL 1995, I.1.7), and $\underline{\alpha}$ is the unique element of Δ which projects onto α then $(M_1)_\alpha$ is also defined as the standard Levi subgroup of G such that $\Delta^{(M_1)_\alpha} = \{\underline{\alpha}\} \cup \Delta^{M_1}$. Then M_1 is a maximal proper Levi subgroup of $(M_1)_\alpha$. Writing $W^{(M_1)_\alpha}_{\text{std}}(M_1)$ for the analogue of $W_{\text{std}}(M_1)$ when replacing G by $(M_1)_\alpha$, we easily show that $W^{(M_1)_\alpha}(M_1)$ has two elements : the identity and the element $s_\alpha = \widetilde{w_0^{M_1}}\widetilde{w_0^{(M_1)_\alpha}}$, where $\widetilde{w_0^{M_1}}$ and $\widetilde{w_0^{(M_1)_\alpha}}$ are the elements of greatest length of $W^{M_1}_{\text{std}}$ and $W^{(M_1)_\alpha}_{\text{std}}$, respectively. Note that $W^{(M_1)_\alpha}_{\text{std}}(M_1)$ embeds in $W_{\text{std}}(M_1)$. This defines an element s_α which is called an *elementary symmetry*.

These elementary symmetries occur in the following Theorem 8 (but this theorem holds for s_α being not necessarily an elementary symmetry, i.e when α is not simple and $(M_1)_\alpha$ not necessarily standard).

Finally, we denote $W(M_1)$ the subset of $W_{\text{std}}(M_1)$ constituted of the set of representatives in W of elements in the quotient group $\{w \in W | w^{-1}M_1w = M_1\} / W^{M_1}$ of minimal length in their right classes modulo W^{M_1}.

1.2.1. The μ function

Harish-Chandra's μ-function is the main ingredient of the Plancherel density for a p-adic reductive group G WALDSPURGER 2003. It assigns to every discrete series representation of a Levi subgroup a complex number and can be analytically extended to a meromorphic function on the space of essentially square-integrable representations of Levi subgroups.

Let $Q = NV$ be a parabolic subgroup of a connected reductive group G over F and σ an irreducible unitary cuspidal representation of N, then the Harish-Chandra's μ-function μ^G corresponding to G defines a meromorphic function $a^*_{N,\mathbb{C}} \to \mathbb{C}$, $\lambda \to \mu^G(\sigma_\lambda)$ (cf. HEIERMANN 2004, Proposition 4.1, SILBERGER 1980b, 1.6) which (in a certain context, see Proposition 4.1 in HEIERMANN 2004) can be written :

$$\mu^G(\sigma_\lambda) = f(\lambda) \prod_{\alpha\in\Sigma(Q)} \frac{(1-q^{\langle\check\alpha,\lambda\rangle})(1-q^{-\langle\check\alpha,\lambda\rangle})}{(1-q^{\epsilon_\alpha+\langle\check\alpha,\lambda\rangle})(1-q^{\epsilon_\alpha-\langle\check\alpha,\lambda\rangle})}$$

where f is a meromorphic function without poles and zeroes on a_N^* and the ϵ_α are non-negative rational numbers such that $\epsilon_\alpha = \epsilon_{\alpha'}$ if α and α' are conjugate. We refer the reader to Sections IV.3 and V.2 of WALDSPURGER 2003 for some further properties of the Harish-Chandra μ function.

Clearly the μ function denoted above μ^G can be defined with respect to any reductive group G, in particular we will use below the functions μ^M for a Levi subgroup M.

In HEIERMANN 2006 and HEIERMANN 2011, with the notations introduced in the Section 1.4.2, the following results are mentioned : Let $P_1 = M_1U_1$ be a standard parabolic subgroup.

Theorem 8 (Harish-Chandra, see HEIERMANN 2011, 1.2)**.** *Fix a root $\alpha\in\Sigma(P_1)$ and an irreducible cuspidal representation σ of M_1.*

a) If $\mu^{(M_1)_\alpha}(\sigma) = 0$ then there exists a unique (see Casselman's notes, 7.1 in « Introduction to the theory of admissible representations of p-adic reductive groups ») non trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ so that $s_\alpha(P_1\cap(M_1)_\alpha) = \overline{P_1}\cap(M_1)_\alpha$ and $s_\alpha\sigma\cong\sigma$.

b) If there exists a unique non trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ so $s_\alpha(P_1\cap(M_1)_\alpha) = \overline{P_1}\cap(M_1)_\alpha$ and $s_\alpha\sigma\cong\sigma$. Then $\mu^{(M_1)_\alpha}(\sigma)\neq 0 \Leftrightarrow I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma)$ is reducible.

If it is reducible, it is the direct sum of two non isomorphic representations.

Where the μ function's factor in this setting is :

$$\mu^{(M_1)_\beta}(\sigma_\lambda) = c_\beta(\lambda).\frac{(1-q^{\langle\check\beta,\lambda\rangle})(1-q^{-\langle\check\beta,\lambda\rangle})}{(1-q^{\epsilon_{\check\beta}+\langle\check\beta,\lambda\rangle})(1-q^{\epsilon_{\check\beta}-\langle\check\beta,\lambda\rangle})}$$

Lemma 9 (Lemma 1.8 in HEIERMANN 2011)**.** *Let $\alpha\in\Delta_\sigma$, $s = s_\alpha$ and assume $(M_1)_\alpha$ is a standard Levi subgroup of G. The operator $J_{sP_1|P_1}$ are meromorphic functions in σ_λ for σ unitary cuspidal representation and λ a parameter in $(a_{M_1}^{(M_1)_\alpha}*)$.*

The poles of $J_{sP_1|P_1}$ are precisely the zeroes of $\mu^{(M_1)_\alpha}$. Any pole has order one and its residu is bijective. Furthermore, $J_{P_1|sP_1}J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\alpha})^{-1}$ up to a multiplicative constant.

Let us summarize the different cases :

— If $\mu^{(M_1)_\alpha}$ has a pole at σ_λ ; then, the operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ (which are necessarily both non-zero) cannot be bijective. Indeed, at σ_λ their product is zero, if any was bijective, it would imply the other is zero.

— If $\mu^{(M_1)_\alpha}$ has a zero in σ_λ; it is Lemma 9 above.

Further by a general result concerning the μ function, it has one and only one pole on the positive real axis if and only if, for σ a unitary irreducible cuspidal representation, $\mu(\sigma) = 0$. Therefore for each $\alpha \in \Sigma_\sigma$, by definition, there will be one λ on the positive real axis such that $\mu^{(M_1)_\alpha}$ has a pole.

Example 1. Consider the group $G = GL_{2n}$ and one of its maximal Levi subgroups $M := GL_n \times GL_n$. Set $\sigma_s := \rho|\det|^s \otimes \rho|\det|^{-s}$ with ρ irreducible unitary cuspidal representation of GL_n. Then, $\mu(\rho \otimes \rho) = 0$ and it is well known that at $s = \pm 1/2$, $\mu(\sigma_s)$ has a pole and the operators $J_{P|\overline{P}}$ and $J_{\overline{P}|P}$ are not bijective.

1.2.2. Standard module induced from a maximal parabolic subgroup

Let $\Theta = \Delta - \{\alpha\}$ for α in Δ, and let $P = P_\Theta$ be a maximal parabolic subgroup of G. We denote ρ_P the half sum of positive roots in U, and for α the unique simple root for G which is not a root for M,

$$\tilde{\alpha} = \frac{\rho_P}{\langle \rho_P, \alpha \rangle}$$

(Rather than $\tilde{\alpha}$, in the split case, we could also take the fundamental weight corresponding to α).

Since ν is in a_M^* (of dimension rank(G) - rank(M)= 1 since M is maximal), and should satisfy $\left\langle \nu, \check{\beta} \right\rangle > 0$ for all $\beta \in \Delta - \Theta = \{\alpha\}$, the standard module in this case is $I_P^G(\tau_{s\tilde{\alpha}})$ where $s \in \mathbb{R}$ such that $s > 0$, and τ is an irreducible tempered representation of M.

1.2.3. Some results on residual points

Let Q be any parabolic subgroup of G, with Levi decomposition $Q = LU$. We recall that the parabolic rank of G (with respect to L) is $rk_{ss}(G) - rk_{ss}(L)$, where rk_{ss} stands for the semi-simple rank. The following definition will be useful :

Definition 10 (residual point)**.** A point σ_ν for σ an irreducible unitary cuspidal representation of L is called a residual point for μ^G if

$$|\{\alpha \in \Sigma(Q) | \langle \check{\alpha}, \nu \rangle = \pm \epsilon_\alpha\}| - 2|\{\alpha \in \Sigma(Q) | \langle \check{\alpha}, \nu \rangle = 0\}| = \dim(a_L^*/a_G^*) = rk_{ss}(G) - rk_{ss}(L)$$

where ϵ_α appears in the Section 1.2.1.

Remark 1. Since the μ function depends only on a complex variable identified with $\sigma \otimes \chi_\lambda$, for $\lambda \in (a_L^G)^*$; once the unitary cuspidal representation σ is fixed we will freely talk about λ (rather than σ_λ) as a residual point.

The main result of Heiermann in HEIERMANN 2004 is the following :

Theorem 11 (Corollary 8.7 in HEIERMANN 2004). *Let $Q = LU$ be a parabolic subgroup of G, σ a unitary cuspidal representation of L, and ν in a_L^*. For the induced representation $I_Q^G(\sigma_\nu)$ to have a discrete series subquotient, it is necessary and sufficient for σ_ν to be a residual point for μ^G and the restriction of σ_ν to A_G (the maximal split component in the center of G) to be a unitary character.* [2]

We will also make a crucial use of the following result from HEIERMANN et OPDAM 2009 :

Proposition 12 (Proposition 2.5 in HEIERMANN et OPDAM 2009). *Let π be a generic representation which is a discrete series of G. There exists a standard parabolic subgroup $Q = LU$ of G and a unitary generic cuspidal representation (σ, E) of L, with $\nu \in \overline{(a_L^*)^+}$ such that π is a subrepresentation of $I_Q^G(\sigma_\nu)$.*

1.2.4. Some results on standard modules

We recall the Langlands' classification (see for instance BOREL et WALLACH 1999 Theorem 2.11 or KONNO 2003)

Theorem 13 (Langlands' classification).
1. *Let $P = MU$ be a standard parabolic subgroup of G, τ (the equivalent class of) an irreducible tempered representation of M and $\nu \in a_M^{+*}$. Then the induced representation $I_P^G(\tau_\nu)$ has a unique irreducible quotient, the Langlands quotient denoted $J(P, \nu, \tau)$*
2. *Let π be an irreducible admissible representation of G. Then there exists a unique triple (P, ν, τ) as in (1) such that $\pi = J(P, \nu, \tau)$. We call this triple the Langlands data, and ν will be called the Langlands parameter of π.*

Theorem 14 (*Standard module conjecture* proved in HEIERMANN et MUIC 2006 and HEIERMANN et OPDAM 2009). *Let $\nu \in a_M^{*+}$, and τ be an irreducible tempered generic representation of M. Denote $J(\tau, \nu)$ the Langlands quotient of the induced representation $I_P^G(\tau_\nu)$. Then, the representation $J(\tau, \nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.*

1.3. Setting and first results

Following HEIERMANN et OPDAM 2009, let us denote $a_{M_1}^{M*} = \mathbb{R}\Sigma^M \subset a_{M_1}^{G*}$, where Σ^M are the roots in Σ which are in M (with basis Δ^M) (see also RENARD 2010 V.3.13).

With the setting of Section 1.2.2, we consider τ a ψ-generic discrete series of M. By the above proposition (Proposition 12) there exists a standard parabolic

2. Alternatively the last condition can be stated as : the projection of ν on a_G^* is zero.

subgroup $P_1 = M_1U_1$ of G, and we could further assume $M_1 \subset M$, σ_ν a cuspidal representation of M_1, Levi subgroup of $M \cap P_1$ such that τ is a generic discrete series that appears as subrepresentation of $I_{M\cap P_1}^M(\sigma_\nu)$, with ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$. Moreover, σ_ν is a residual point for μ^M.

By transitivity of induction, we have :

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_P^G(I_{M\cap P_1}^M(\sigma_\nu))_{s\tilde{\alpha}} = I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$$

where $s \in \mathbb{R}$ satisfies $s > 0$ and $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$ (Rather than $\tilde{\alpha}$, we could also take the fundamental weight corresponding to α, but we will rather follow a convention of Shahidi [see CASSELMAN et SHAHIDI 1998]).

Remark 2. The reader should note that our standard module $I_P^G(\tau_{s\tilde{\alpha}})$ is induced from an essentially square integrable representation $\tau_{s\tilde{\alpha}}$. The general case of a tempered representation τ will follow in the Corollary 63. Throughout this paper, we will adopt the following convention : τ will denote a discrete series representation, σ an (irreducible) cuspidal representation. Also following notations (as for instance in HANZER 2010 or MOEGLIN et TADIC 2002), $\pi \leq \Pi$ means π is realised as a subquotient of Π, whereas $\pi \hookrightarrow \Pi$ is stronger, and means it embeds as a subrepresentation.

In the following sections we will study the generic subquotient of $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and consider the cases where either there exists a discrete series subquotient, or there isn't and therefore tempered or non-tempered generic (not square integrable) subquotients may occur.

Given a generic discrete series subquotient γ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$, using Proposition 12 above, it appears as a generic subrepresentation in some induced representation $I_{P'}^G(\sigma'_{\lambda'})$ for λ' in the closure of the positive Weyl chamber with respect to P', and σ' irreducible cuspidal generic.

The set-up is summarized in the following diagram :

$$\begin{tikzcd}
\gamma \leq & I_P^G(\tau_{s\tilde{\alpha}}) \arrow[r] & I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}}) \\
\gamma \arrow[rr, hook] & & I_{P'}^G(\sigma'_{\lambda'}) \arrow[u, dotted]
\end{tikzcd}$$

We will investigate the existence of a bijective up-arrow on the right of this diagram.

1.4. Intertwining operators

1.4.1.

Assume the existence of a common generic subquotient in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and $I_{P'}^G(\sigma'_{\lambda'})$.

We would like to construct a bijective operator between these two induced modules in order to transfer the irreducible subrepresentations in $I_{P'}^G(\sigma'_{\lambda'})$ to $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$. The existence of this bijective operator is the content of the following proposition.

Lemma 15. *Let P_1 and Q be two parabolic subgroups of G having the same Levi subgroup M_1.*

Then there exist an isomorphism $r_{P_1|Q}$ between the two induced modules $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$ for any irreducible unitary cuspidal representation σ whenever λ is dominant for both P_1 and Q.

Proof. We first assume that Q and P_1 are adjacent [3]. We denote β the common root of $\Sigma(\overline{Q})$ and $\Sigma(P_1)$. $\overline{Q}$ is the parabolic subgroup opposite to Q with Levi subgroup M_1.
We have

$$I_Q^G(\sigma_\lambda) = I_{Q_\beta}^G(I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda))$$

where $(M_1)_\beta$ is the centralizer of A_β (the identity component in the kernel of β) in G, a semi-standard Levi subgroup (confer section 1 in Waldspurger 2003), and the same inductive formula holds replacing Q by P_1.
Since λ is dominant for both Q and P_1, $\langle\lambda,\beta\rangle \geq 0$ (since β is a root in $\Sigma(P_1)$), but also $\langle\lambda,-\beta\rangle \geq 0$ since $-\beta$ is a root in $\Sigma(Q)$. Therefore $\left\langle\check{\beta},\lambda\right\rangle = 0$.
We have λ in $a_{M_1}^*$ which decomposes as

$$(a_{M_1}^{(M_1)_\beta})^* \oplus (a_{(M_1)_\beta})^*$$

and we write $\lambda = \mu \oplus \eta$. The dual of the Lie algebra, $(a_{M_1}^{(M_1)_\beta})^*$, is of dimension one (since M_1 is a maximal Levi subgroup in $(M_1)_\beta$) generated by $\check{\beta}$. If $\left\langle\check{\beta},\lambda\right\rangle = 0$, the projection of λ on $(a_{M_1}^{(M_1)_\beta})^*$ is also zero. That is $\left\langle\check{\beta},\mu\right\rangle = 0$ or χ_μ is unitary.

Therefore with σ unitary, and χ_μ a unitary character, the representations

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu) \qquad \text{and} \qquad I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu)$$

are unitary. Since they trivially satisfy the conditions (i) of Theorem 2.9 in Bernstein et Zelevinsky 1977 (see also Renard 2010 VI.5.4) they have equivalent

3. Two parabolic subgroups Q and P_1 are adjacent along α if $\Sigma(P_1)\cap -\Sigma(Q) = \{\alpha\}$

Jordan-Hölder composition series, and are therefore isomorphic (As unitary representations, having equivalent Jordan-Hölder composition series). Tensoring with χ_η preserves the isomorphism between

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu) \qquad \text{and} \qquad I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu)$$

That is, there exist an isomorphism between $I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and $I_{P\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$. The induction of this isomorphism therefore gives an isomorphism between $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$ that we call $r_{P_1|Q}$.

If we further assume that Q and P_1 are not adjacent, but can be connected by a sequence of adjacent parabolic subgroups of G,

$$\{Q = Q_1, Q_2, Q_3, \ldots, Q_n = P_1\}$$

with

$$\Sigma(Q_i) \cap \Sigma(\overline{Q_{i+1}}) = \{\beta_i\}$$

We have the following set-up :

$$I_Q^G(\sigma_\lambda) \overset{r_{Q_2|Q}}{\rightarrow} I_{Q_2}^G(\sigma_\lambda) \overset{r_{Q_3|Q_2}}{\rightarrow} I_{Q_3}^G(\sigma_\lambda) \ldots \overset{r_{Q_n|Q_{n-1}}}{\rightarrow} I_{P_1}^G(\sigma_\lambda)$$

Again, under the assumption that λ is dominant for P_1 and Q, we have $\langle \beta_i, \lambda\rangle \geq 0$ and $\langle -\beta_i, \lambda\rangle \geq 0$ for each β_i in $\Sigma(P_1) \cap \Sigma(\overline{Q})$, hence $\left\langle \check{\beta}_i, \lambda \right\rangle = 0$. Therefore there exists an isomorphism between $I_{Q_i}^G(\sigma_\lambda)$ and $I_{Q_{i+1}}^G(\sigma_\lambda)$ denoted $r_{Q_{i+1}|Q_i}$.
The composition of the isomorphisms $r_{Q_{i+1}|Q_i}$ will eventually give us the desired isomorphism between $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$. □

Proposition 16. *Let $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$ be two induced modules with σ (resp.σ') irreducible cuspidal representation of M_1 (resp M'), $\lambda \in a_{M_1}^*$, $\lambda' \in a_{M'}^*$, sharing a common subquotient, then :*

1. *There exists an element g in G such that ${}^gP' := gP'g^{-1}$ and P_1 have the same Levi subgroup.*
2. *If λ and λ' are dominant for P_1 (resp. P'), there exists an isomorphism R_g between $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$*

Proof. First, since the representations $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$ share a common subquotient by Theorem 2.9 in Bernstein et Zelevinsky 1977, there exist an element g in G such that $M_1 = gM'g^{-1}$, ${}^g\sigma'_{\lambda'} = \sigma_\lambda$ and $g\lambda' = \lambda$, where ${}^g\sigma(x) = \sigma(g^{-1}xg)$ for $x \in M_1$.
The last point follows from the equality ${}^g\chi_{\lambda'} = \chi_{g\lambda'}$.

For the second point, we first apply the map $t(g)$ between $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{{}^gP'}^G({}^g\sigma'_{\lambda'})$ which is an isomorphism that sends f on $f(g^{-1}.)$

As λ' is dominant for P', $g\lambda' = \lambda$ is dominant for ${}^gP'$, and we can further apply the isomorphism defined in the previous lemma (Lemma 15) : $r_{P_1|\,{}^gP'}(\sigma_\lambda)$ (Since

P_1 and ${}^gP'$ have the same Levi subgroup : M_1), we will therefore have :

$$I_{P'}^G(\sigma'_{\lambda'}) \overset{t(g)}{\to} I_{{}^gP'}^G({}^g\sigma', g.\lambda') \overset{r_{P_1|{}^gP'}}{\to} I_{P_1}^G(\sigma_\lambda)$$

and R_g is the isomorphism given by the composition of $t(g)$ and $r_{P_1|{}^gP'}$. □

1.4.2. Intertwining operators with non-generic kernels

Definition 17. A set of Langlands data for G is a triple (P, τ, ν) with the following properties :

1. $P = MU$ is a standard parabolic subgroup of G
2. ν is in $(a_M^*)^+$
3. τ is (the equivalence class of) an irreducible tempered representation of M.

Our objective is to embed an irreducible generic subquotient as a subrepresentation in an induced module from data (P_1, σ, λ)[4] knowning it embeds in one with Langlands' data (P', σ', λ'). If the intertwining operator between those two induced modules has non-generic kernel, the generic subrepresentation will necessarily appear in the image of the intertwining operator, and therefore will appear as a *subrepresentation* in the induced module with Langlands' data (P_1, σ, λ). We detail the conditions to obtain the non-genericity of the kernel of the intertwining operator.

Proposition 18. *Let P_1 and Q be two parabolic subgroups of G having the same Levi subgroup M_1.*

Consider the two induced modules $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$, and assume σ is an irreducible generic cuspidal representation and λ is dominant for P_1 and anti-dominant for Q. Then there exists an intertwining map from $I_Q^G(\sigma_\lambda)$ to $I_{P_1}^G(\sigma_\lambda)$ which has non-generic kernel.

Proof. We first assume that Q and P_1 are adjacent. We denote β the common root of $\Sigma(Q)$ and $\Sigma(\overline{P_1})$.

We have $I_Q^G(\sigma_\lambda) = I_{Q_\beta}^G(I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda))$ where $(M_1)_\beta$ is the centralizer of A_β (the identity component in the kernel of β) in G, a semi-standard Levi subgroup (confer Section 1 in Waldspurger 2003), and the same inductive formula holds replacing Q by P_1. Then, there are two cases : The case of $\left\langle \check{\beta}, \lambda \right\rangle = 0$ is Lemma 15. If $\left\langle \check{\beta}, \lambda \right\rangle > 0$, let us consider the intertwining operator defined in Section 1.2 between $I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and $I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and assume it is not an isomorphism. The representation σ being cuspidal, these modules are length two representations by the Corollary 7.1.2 of Casselman's « Introduction to the theory of admissible

4. This is not necessarily a Langlands data since as explained in the beginning of Section 1.5 the parameter λ is not necessarily in the positive Weyl chamber $(a_{M_1}^*)^+$

representations of p-adic reductive groups ». Let S be the kernel of this intertwining map and the Langlands quotient $J(\sigma, P_1 \cap (M_1)_\beta, \lambda)$ its image. One has the exact sequences :

$$0 \to S \to I^{(M_1)_\beta}_{P_1 \cap (M_1)_\beta}(\sigma_\lambda) \to J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \to 0$$

$$0 \to J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \to I^{(M_1)_\beta}_{Q \cap (M_1)_\beta}(\sigma_\lambda) \to S \to 0$$

Further, the projection from

$$I^{(M_1)_\beta}_{Q \cap (M_1)_\beta}(\sigma_\lambda)$$

to

$$I^{(M_1)_\beta}_{Q \cap (M_1)_\beta}(\sigma_\lambda)/J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \cong S \subset I^{(M_1)_\beta}_{P_1 \cap (M_1)_\beta}(\sigma_\lambda)$$

defines a map whose kernel, $J(\sigma, P_1 \cap (M_1)_\beta, \lambda)$, is not generic (by the main result of Heiermann et Muic 2006 which proves the Standard module Conjecture). In other words, we have the following exact sequence :

$$0 \to J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \to I^{(M_1)_\beta}_{Q \cap (M_1)_\beta}(\sigma_\lambda) \xrightarrow{A} I^{(M_1)_\beta}_{P_1 \cap (M_1)_\beta}(\sigma_\lambda)$$

Inducing from $(P_1)_\beta$ to G, one observes that the kernel of the induced map $(I^G_{(P_1)_\beta}(A))$ is the induction of the kernel $J(\sigma, P_1 \cap (M_1)_\beta, \lambda)$. Therefore the kernel of the induced map is non-generic (here, we use the fact that there exists an isomorphism between the Whittaker models of the inducing and the induced representations, using result of Rodier 1972 and Casselman et Shalika 1980).

Assume now that Q and P_1 are not adjacent, but can be connected by a sequence of adjacent parabolic subgroups of G,

$$\{Q = Q_1, Q_2, Q_3, \ldots, Q_n = P_1\}$$

with

$$\Sigma(Q_i) \cap \Sigma(\overline{Q_{i+1}}) = \{\beta_i\}$$

We have the following set-up :

$$I^G_Q(\sigma_\lambda) \overset{r_{Q_2|Q}}{\rightarrow} I^G_{Q_2}(\sigma_\lambda) \overset{r_{Q_3|Q_2}}{\rightarrow} I^G_{Q_3}(\sigma_\lambda) \ldots \overset{r_{Q_n|Q_{n-1}}}{\rightarrow} I^G_{P_1}(\sigma_\lambda)$$

Assume that certain maps $r_{Q_{i+1}|Q_i}$ have a kernel, by the same argument as above their kernels are non-generic and therefore the kernel of the composite map is non-generic. Indeed, we have the next Lemma 19. □

Lemma 19. *The composition of operators with non-generic kernel has non-generic kernel.*

Proof. Consider first the composition of two operators, A and B as follows :

$$I^G_Q(\sigma_\lambda) \xrightarrow{A} I^G_{Q_2}(\sigma_\lambda) \xrightarrow{B} I^G_{P_1}(\sigma_\lambda)$$

Clearly, the kernel of the composite $(B \circ A)$ contains the kernel of A and the elements in the space of the representation $I_Q^G(\sigma_\lambda)$, x, such that $A(x)$ is in the kernel of B.

This means we have the following sequence of homomorphisms :

$$0 \to \ker(A) \to \ker(B \circ A) \xrightarrow{A} \ker(B) \cap \mathrm{Im}(A) \to 0$$

pull-back by A^{-1} of element in $Ker(B)$. The pull-back of a non-generic kernel yields a non-generic subspace in the pre-image. The fact that this sequence is exact is clear except for the surjectivity of the map $\ker(B \circ A) \xrightarrow{A} \ker(B) \cap \mathrm{Im}(A)$. But, if $y \in \ker(B) \cap \mathrm{Im}(A)$, then there exists x such that $A(x) = y$ and we have $B \circ A(x) = B(y) = 0$ since $y \in \ker(B)$.

If both $\ker(B)$ and $\ker(A)$ are non-generic, the kernel of $(B \circ A)$ is itself non-generic. Extending the reasoning to a sequence of rank one operators with non-generic kernels yields the result. $\square$

We have observed that the nature of intertwining operators rely on the dominance of the parameters λ and λ'. We now need a more explicit description of these parameters ; to do so we will call on a result first presented in Opdam 2004 in the Hecke algebra context (Theorem 103 in Appendix F) and further developed in Heiermann 2006.

1.5. Description of residual points via Bala-Carter

With the notations of Section 1.3, we will study generic subquotient in induced modules $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and $I_{P'}^G(\sigma'_{\lambda'})$.

One needs to observe, following the construction of our setting in Section 1.3, that ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$, whereas $s\tilde{\alpha}$ is in the positive Weyl chamber $(a_M^*)^+$, therefore it is not expected that $\nu + s\tilde{\alpha}$ should be in the closure of the positive Weyl chamber $\overline{(a_{M_1}^*)^+}$. This is explained in the Appendix C.

In particular, let α be the only root in $\Sigma(A_0)$ which is not in Lie(M), we may have $\langle \nu, \check{\alpha} \rangle < 0$ and therefore for some roots $\beta \in \Sigma(A_{M_1})$, written as linear combination containing the simple root α, we may also have : $\left\langle \nu + s\tilde{\alpha}, \check{\beta} \right\rangle < 0$.

However, by the result presented in Appendix F, if $\nu + s\tilde{\alpha}$ is a residual point, it is in the Weyl group orbit of a dominant residual point (i.e. one whose expression can be directly deduced from a weighted Dynkin diagram). We therefore define :

Definition 20 (dominant residual point)**.** A residual point σ_λ for σ an irreducible cuspidal representation is dominant if λ is in the closed positive Weyl chamber $\overline{(a_M^*)^+}$.

Bala-Carter theory allows to describe explicitely the Weyl group orbit of a residual point. In the context of reductive p-adic groups studied in Heiermann 2006 (see in particular Proposition 6.2 in Heiermann 2006), the fact that σ_λ lies in the cuspidal support of a discrete series can be translated somehow to the assertion that σ_λ corresponds to a distinguished nilpotent orbit in the dual of the Lie algebra ${}^L\mathfrak{g}$, and therefore by Proposition 102 (see also 103) in Appendix F to a weighted Dynkin diagram. [5]

In the present work we treat the case of weighted Dynkin diagrams of type A, B, C, D. The key proposition is Proposition 24 below.

Our setting

Recall that in Section 1.3 we embedded the standard module as follows :

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_P^G(I_{M\cap P_1}^M(\sigma_\nu))_{s\tilde{\alpha}} = I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$$

By hypothesis, σ_ν is a residual point for μ^M.

$\lambda = \nu + s\tilde{\alpha}$ is in $a_{M_1}^*$.

Describing explicitely the form of the parameter $\lambda \in a_{M_1}^*$ is essential for two reasons : first, to determine the nature (i.e discrete series, tempered, or non-tempered representations) of the irreducible generic subquotients in the induced module $I_{P_1}^G(\sigma_\lambda)$; secondly, to describe the intertwining operators and in particular the (non)-genericity of their kernels.

We will explain the following correspondences :

$$\begin{aligned}&\{\text{dominant residual point}\} \leftrightarrow \{\text{Weighted Dynkin diagram}\}\\ &\qquad \leftrightarrow \{\text{residual segments}\} \leftrightarrow \{\text{Jumps of the residual segment}\} \qquad (1.1)\end{aligned}$$

The connection between residual points and roots systems involved for Weighted Dynkin Diagrams require a careful description of the involved participants :

The root system

Let us now recall that $W(M_1)$ the set of representatives in W of elements in the quotient group $\{w \in W | w^{-1}M_1 w = M_1\} / W^{M_1}$ of minimal length in their right classes modulo W^{M_1}.

Assume σ is a unitary cuspidal representation of a Levi subgroup M_1 in G, and let $W(\sigma, M_1)$ be the subgroup of $W(M_1)$ stabilizer of σ. The Weyl group of Σ_σ is W_σ, the subgroup of $W(M_1, \sigma)$ generated by the reflexions s_α.

Proposition 21 (3.5 in Silberger 1981). *The set* $\Sigma_\sigma := \left\{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$ *is a root system.*

5. Notice that Proposition 102 requires : G to be a semi-simple adjoint group ; a certain parameter k_α to equal one for any root α in Φ ; further, it concerns only the case of unramified characters.

For $\alpha \in \Sigma_\sigma$, let s_α the unique element in $W^{(M_1)_\alpha}(\sigma)$ which conjugates $P_1 \cap M_\alpha$ and $\overline{P_1} \cap (M_1)_\alpha$. The Weyl group W_σ of Σ_σ identifies to the subgroup of $W(M_1, \sigma)$ generated by reflexions s_α, $\alpha \in \Sigma_\sigma$.

$\check{\alpha}$ the unique element in $a_{M_1}^{(M_1)_\alpha}$ which satisfies $\langle \check{\alpha}, \alpha \rangle = 2$.

Then $\Sigma_\sigma^\vee := \{\check{\alpha} | \alpha \in \Sigma_\sigma\}$ is the set of coroots of Σ_σ, the duality being that of a_{M_1} and $a_{M_1}^*$.

The set $\Sigma(P_1) \cap \Sigma_\sigma$ is the set of positive roots for a certain order on Σ_σ.

Remark 3. An equivalent proposition is proved in Heiermann 2011 (Proposition 1.3). There, the author considers $\mathcal{O}$ the set of equivalence classes of representations of the form $\sigma \otimes \chi$ where χ is an unramified character of M_1. He proves that the set $\Sigma_{\mathcal{O},\mu} := \left\{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha} \text{has a zero on } \mathcal{O}\right\}$ is a root system.

The Weyl group of G relative to a maximal split torus in M_1 acts on $\mathcal{O}$. The previous statement holds replacing W_σ by $W(M_1, \mathcal{O})$, the subgroup of $W(M_1)$ stabilizer of $\mathcal{O}$.

Lemma 22. *If σ is the trivial representation of $M_1 = M_0$ and λ is in the Weyl chamber a_0^*, the root system Σ_σ is the root system of the group G relative to A_0 (with length given by the choice of P_0).*

Proof. Recall that

$$\Sigma_\sigma := \left\{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$$

is a root system.

We now apply this definition to the trivial representation. Clearly, for any $\alpha \in \Sigma(A_0)$, the trivial representation is fixed by any element in $W^{(M_0)_\alpha}(M_0)$, and therefore by s_α satisfying $s_\alpha(P_0 \cap (M_0)_\alpha) = \overline{P_0} \cap (M_0)_\alpha$.

It is well-known that the induced representation $I_{P_0 \cap (M_0)_\alpha}^{(M_0)_\alpha}(\mathbf{1})$ is irreducible; therefore using Harish Chandra's Theorem (Theorem 8) above, $\mu^{(M_0)_\alpha}(\mathbf{1}) = 0$. Then

$$\left\{\alpha \in \Sigma_{\text{red}}(A_0) | \mu^{(M_0)_\alpha}(\mathbf{1}) = 0\right\} := \left\{\alpha \in \Sigma(A_0) | \mu^{(M_0)_\alpha}(\mathbf{1}) = 0\right\} = \{\alpha \in \Sigma(A_0)\}\,.$$

$\square$

In general, the root system Σ_σ is the disjoint union of irreducible or empty components $\Sigma_{\sigma,i}$ for $i = 1, \ldots, r$. This will be detailed in the Subsection 1.5.4.2.

Proposition 23. *Let G be a quasi-split group whose root system Σ is of type A, B, C or D. Then the irreducible components of Σ_σ are of type A, B, C or D.*

Proof. See the Appendix E $\square$

How the root system Σ_σ determines the Weighted Dynkin diagrams to be used in this work

Proposition 24. *Assume G quasi-split over F. Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1. Put $\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$. Let*

$$d = rk_{ss}(G) - rk_{ss}(M_1).$$

The set Σ_σ is a root system in a subspace of $a_{M_1}^*$ (cf. Silberger in SILBERGER *1981 3.5). Suppose that the irreducible components of Σ_σ are all of type A, B, C or D. Denote, for each irreducible component $\Sigma_{\sigma,i}$ of Σ_σ, by $a_{M_1}^{M^i *}$ the subspace of $a_{M_1}^{G*}$ generated by $\Sigma_{\sigma,i}$, by d_i its dimension and by $e_{i,1}, \dots, e_{i,d_i}$ a basis of $a_{M_1}^{M^i *}$ (resp. of a vector space of dimension $d_i + 1$ containing $a_{M_1}^{M^i *}$ if $\Sigma_{\sigma,i}$ is of type A) so that the elements of the root system $\Sigma_{\sigma,i}$ are written in this basis as in Bourbaki* Groupes et Algèbres de Lie, Chapitre 4,5, et 6.

For each i, there is a unique real number $t_i > 0$ such that, if $\alpha = \pm e_{i,j} \pm e_{i,j'}$ lies in $\Sigma_{\sigma,i}$, then $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\frac{t_i}{2}(\pm e_{i,j} \pm e_{i,j'})})$ is reducible.

If $\Sigma_{\sigma,i}$ is of type B or C, then there is in addition a unique element $\epsilon_i \in \{1/2, 1\}$ such that $I_{P_1 \cap (M_1)_{\alpha_{i,d_i}}}^{(M_1)_{\alpha_{i,d_i}}}(\sigma_{\epsilon_i t_i e_{i,d_i}})$ is reducible.

Let $\lambda = \sum_i \sum_{j=1}^{d_i} \lambda_{i,j} e_{i,j}$ be in $\overline{a_{M_1}^{G*+}}$ with $\lambda_{i,j}$ real numbers.

Then σ_λ is in the cuspidal support of a discrete series representation of G, if and only if the following two properties are satisfied

(i) $d = \sum_i d_i$;

(ii) For all i, $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic of a simple complex adjoint group of

- type D_{d_i} (resp. A_{d_i}) if $\Sigma_{\sigma,i}$ is of type D (resp. A) ;

otherwise :

- of type C_{d_i}, if $\epsilon_i = 1/2$;

- of type B_{d_i}, if $\epsilon_i = 1$.

Proof. As λ lies in $a_{M_1}^{G*}$, σ_λ lies in the cuspidal support of a discrete series representation of G, if and only if it is a residual point of Harish-Chandra's μ-function.

Denote $e_{i,j;i',j'}^{\pm}$ the rational character of A_{M_1} whose dual pairing with an element x of $a_{M_1}^G$ with coordinates

$$(x_{1,1}, \dots, x_{1,d_1}, x_{2,1}, \dots, x_{2,d_2}, \dots, x_{r,1}, \dots, x_{r,d_r})$$

in the dual basis equals $x_{i,j} x_{i',j'}^{\pm 1}$ and by $e_{i,j}^{\pm}$ the one whose dual pair equals $x_{i,j}^{\pm 1}$.

The μ-function decomposes as $\prod_{\alpha \in \Sigma(P)} \mu^{M_\alpha}$. By assumption, the function $\lambda \mapsto \mu^{M_\alpha}(\sigma_\lambda)$ won't have a pole or zero on $a_{M_1}^*$ except if $\alpha \in \Sigma_\sigma$. This means that

(i) α is of the form $e_{i,j;i,j'}^{-}$, $j < j'$;

(ii) α is of the form $e_{i,j;i,j'}^{+}$, $j < j'$, and $\Sigma_{\sigma,i}$ of type B, C or D ;

(iii) α is of the form $e_{i,j}^+$ or $2e_{i,j}^+$ and $\Sigma_{\sigma,i}$ of respectively type B or C.

Let $(\lambda_{i,j})_{i,j}$ be a family of real numbers as in the statement of the proposition and put $\lambda = \sum_i \sum_{j=1}^{d_i} \lambda_{i,j} e_{i,j}$. It follows from Langlands-Shahidi theory (cf. the proof of Theorem 5.1 in Heiermann et Opdam 2009) that there is, for each i, a real number $t_i > 0$ and $\epsilon_i \in \{1/2, 1\}$, so that

If $\alpha = e_{i,j;i,j'}^{\pm} \in \Sigma_\sigma$, $j < j'$, then

$$\mu^{M_\alpha}(\sigma_\lambda) = c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}}) \frac{(1-q^{\lambda_{i,j} \pm \lambda_{i,j'}})(1-q^{-\lambda_{i,j} \mp \lambda_{i,j'}})}{(1-q^{t_i - \lambda_{i,j} \pm \lambda_{i,j'}})(1-q^{t_i + \lambda_{i,j} \mp \lambda_{i,j'}})},$$

where $c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}})$ denotes a rational function in $\sigma_{(\lambda_{i,j})_{i,j}}$, which is regular and nonzero for real $\lambda_{i,j}$.

If $\alpha = e_{i,j} \in \Sigma_\sigma$ or $\alpha = 2e_{i,j} \in \Sigma_\sigma$, then

$$\mu^{M_\alpha}(\sigma_{(\lambda_{i,j})_{i,j}}) = c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}}) \frac{(1-q^{\lambda_{i,j}})(1-q^{-\lambda_{i,j}})}{(1-q^{\epsilon_i t_i - \lambda_{i,j}})(1-q^{\epsilon_i t_i + \lambda_{i,j}})}$$

with $\epsilon_i = 1, 1/2$.

Put $\kappa_i^+ = 0$ if $\Sigma_{\sigma,i}$ is of type A and put $\kappa_i = 0$ if $\Sigma_{\sigma,i}$ is of type A or D and otherwise $\kappa_i = \kappa_i^+ = 1$. As λ is in the closure of the positive Weyl chamber, it follows that, for σ_λ to be a residual point of Harish-Chandra's μ-function, it is necessary and sufficient, that for every i, one has

$$d_i = |\{(j,j')|j<j', \lambda_{i,j} - \lambda_{i,j'} = t_i\}| + \kappa_i^+|\{(j,j')|j<j', \lambda_{i,j} + \lambda_{i,j'} = t_i\}| + \kappa_i|\{j|\lambda_{i,j} = \epsilon_i t_i\}| \tag{1.2}$$

$$-2[|\{(j,j')|j<j', \lambda_{i,j} - \lambda_{i,j'} = 0\}| + \kappa_i^+|\{(j,j')|j<j', \lambda_{i,j} + \lambda_{i,j'} = 0\}| + \kappa_i|\{j|\lambda_{i,j} = 0\}|]. \tag{1.3}$$

If $\kappa_i = 0$ or $\epsilon_i = 1$, then this is the condition for $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ defining a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type A_{d_i}, D_{d_i} or B_{d_i} as in 5.7.5 in Carter 1985. If $\epsilon_i = 1/2$, one sees that $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ defines a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type C_{d_i}.

In other words, $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of an adjoint simple complex group of type B_n, C_n or D_n, if $\kappa_i^+ = 1$ and $\kappa_i\epsilon_i$ is respectively 1, 1/2 or 0, and of type A_n if $\kappa_i = 0$.

□

Example 2 (See also Proposition 1.13 in Heiermann 2011 and Proposition 76 in Appendix D)**.** In the context of classical groups, let us spell out the Levi subgroups and cuspidal representations of these Levi considered in the previous proposition :

Let M_1 be a standard Levi subgroup of a classical group G and σ a generic irreducible unitary cuspidal representation of M_1.

Then, up to conjugation by an element of G, we can assume :

$$M_1 = \underbrace{GL_{k_1} \times \ldots GL_{k_1}}_{d_1 \text{ times}} \times \underbrace{GL_{k_2} \times \ldots \times GL_{k_2}}_{d_2 \text{ times}} \times \ldots \times \underbrace{GL_{k_r} \times \ldots \times GL_{k_r}}_{d_r \text{ times}} \times G(k)$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G and

$$\sigma = \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots \otimes \sigma_2 \ldots \ldots \otimes \sigma_r \otimes \ldots \otimes \sigma_r \otimes \sigma_c$$

Let us assume $k \neq 0$, and $\sigma_i \ncong \sigma_j$ if $j \neq i$.

We identify A_{M_1} to $\mathbb{T} = \mathbb{G}_m^{d_1} \times \mathbb{G}_m^{d_2} \times \ldots \times \mathbb{G}_m^{d_r}$ and denote $\alpha_{i,j}$ the rational character of A_{M_1} (identified with $\mathbb{T}$) which sends an element

$$x = (x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}, \ldots, x_{r,1}, \ldots, x_{r,d_r})$$

to $x_{i,j}x_{i,j+1}^{-1}$ if $j < d_i$ and to x_{i,d_i} if $j = d_i$.

Let $(s_{i,j})_{i,j}$ be a family of non-negative real numbers, $1 \leq i \leq r$, $1 \leq j \leq d_i$ and $s_{i,j} \geq s_{i,j+1}$ for i fixed. Then,

$$\sigma_1|\cdot|^{s_{1,1}} \otimes \ldots \sigma_1|\cdot|^{s_{1,d_1}} \otimes \sigma_2|\cdot|^{s_{2,1}} \otimes \ldots \sigma_2|\cdot|^{s_{2,d_2}} \otimes \ldots \otimes \sigma_r|\cdot|^{s_{r,1}} \otimes \ldots \sigma_r|\cdot|^{s_{r,d_r}} \otimes \sigma_c.$$

is in the cuspidal support of a discrete series representations of G, if and only if the following properties are satisfied :

(i) one has $\sigma_i \simeq \sigma_i^\vee$ for every i ;

ii) denote by s_i the unique element in $\{0, 1/2, 1\}$ such that the representation of $G(k + k_i)$ parabolically induced from $\sigma_i|\cdot|^{s_i} \otimes \sigma_c$ is reducible (we use the result of Shahidi on reducibility points for generic cuspidal representations).

Then, for all i, $2(s_{i,1}, \ldots, s_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of a simple complex adjoint group of

- type D_{d_i} if $s_i = 0$; then $\Sigma_{\sigma,i} = \{\alpha_{i,1}, \ldots, \alpha_{i,d_i-1}, \alpha_{i,d_i-1} + 2\alpha_{i,d_i}\}$
- type C_{d_i} if $s_i = 1/2$; then $\Sigma_{\sigma,i} = \{\alpha_{i,1}, \ldots, 2\alpha_{i,d_i}\}$
- type B_{d_i} if $s_i = 1$; then $\Sigma_{\sigma,i} = \{\alpha_{i,1}, \ldots, \alpha_{i,d_i-1}, \alpha_{i,d_i}\}$.

For $i \neq j$, since $\sigma_i \ncong \sigma_j$, we have $\Sigma_{\sigma,i} \neq \Sigma_{\sigma,j}$.

Then M^i is isomorphic to

$$\underbrace{GL_{k_1} \times \ldots GL_{k_1}}_{d_1 \text{ times}} \times \underbrace{GL_{k_2} \times \ldots \times GL_{k_2}}_{d_2 \text{ times}} \times \ldots \times \ldots \times \underbrace{GL_{k_r} \times \ldots \times GL_{k_r}}_{d_r \text{ times}} \times G(k+d_ik_i)$$

1.5.1. From weighted Dynkin diagrams to residual segments

The Dynkin diagram of a distinguished parabolic subgroup mentioned in the Proposition 24 are also called *Weighted Dynkin diagrams* : a definition is given in Appendix F.1 and their forms are given in Appendix B.

Let a parameter $\nu \in a^*_{M_1}$ be written $(\nu_1, \nu_2, \ldots, \nu_n)$ in a basis $\{e_1, e_2, \ldots, e_n\}$ (resp. $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ for type A) (such that this basis is the canonical basis associated to the classical Lie algebra a^*_0, as in *Groupes et Algèbres de Lie, Chapitre 4,5, et 6* when $M_1 = M_0$) and assume it is a dominant residual point. As it is dominant, observe that $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n \geq 0$ (resp $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$ for type A). Further it corresponds by the previous Proposition (24) to a weighted Dynkin diagram of a certain type A, B, C or D (see also Bala-Carter theory presented in Appendix F).

Let us explain the following correspondence :

$$\{\text{Weighted Dynkin diagram}\} \leftrightarrow \{\text{residual segment}\} \tag{1.4}$$

First, let us explain the following assignement :
WDD $\rightarrow \nu$, where ν is the vector with coordinates $\langle \nu, \alpha_i \rangle$.

Let us start with a weighted Dynkin diagram of type A, B, C or D. The weights under roots α_i are 2 (respectively 0) which correspond to $\langle \nu, \alpha_i \rangle = 1$ (respectively 0). (see the weighted Dynkin diagrams given in Appendix B).

Notice that we abusively use α_i rather than $\check{\alpha}_i$ in the product expression, to be consistant with the notations in the weighted Dynkin diagrams.

Using the expressions of α_i in the canonical basis (for instance $\alpha_i = e_i - e_{i+1}$, $2e_i$, or e_i), we compute the vector of coordinates $(\nu_1, \nu_2, \ldots, \nu_n)$ with integers or half-integers entries.

For instance, for $\alpha_i = e_i - e_{i+1}$, when $\langle \nu, \alpha_i \rangle = \langle \sum_{i=1}^n \nu_i e_i, \alpha_i \rangle = 1$, we get $\nu_i - \nu_{i+1} = 1$, whereas if $\langle \nu, \alpha_i \rangle = 0$ then $\nu_i - \nu_{i+1} = 0$.

Conversely, let us be given a vector of coordinates $(\nu_1, \nu_2, \ldots, \nu_n)$ with integers or half-integers entries and the type of root system (A, B, C or D). Using the relations ν_i and ν_{i+1} for any i, we deduce the weights under each root α_i and therefore obtain the weighted Dynkin diagram.

Definition 25 (residual segment)**.** The residual segment of type B, C, D associated to the dominant residual point $\nu := (\nu_1, \nu_2, \ldots, \nu_n) \in \overline{a^{*+}_{M_1}}$ (depending on a fixed irreducible cuspidal representation σ of M_1) is the expression in coordinates of this dominant residual point in the basis of $a^*_{M_1}$ (the basis such that the roots in the Weighted Dynkin diagram are canonically expressed as in *Groupes et Algèbres de Lie, Chapitre 4,5, et 6*).

It is therefore a decreasing sequence of positive (half)-integers uniquely obtained from a Weighted Dynkin diagram by the aformentioned procedure.

It is uniquely characterized by :

- An infinite tuple $(\ldots, 0, n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_0)$ or $(\ldots, 0, n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_{1/2})$ where n_i is the number of times the integer or half-integer value i appears in the sequence.
- The greatest (half)-integer in the sequence, ℓ, such that $n_\ell = 1, n_{\ell-1} = 2$ if it exists.

— the greatest integer, m, such that, for any $i \in \{1, \ldots, m\}$, $n_{\ell+i} = 1$ and for any $i > m$, $n_{\ell+i} = 0$.

This residual segment uniquely determines the weighted Dynkin diagram of type B, C or D from which it originates.

Therefore the values obtained for the n_i's depend on the Weighted Dynkin diagram (see the Appendix B) one observes the following relations :

— Type B : $n_\ell = 1, n_{\ell-1} = 2$, $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 = \frac{n_1 - 1}{2}$ if n_1 is odd or $n_0 = \frac{n_1}{2}$ if n_1 is even. (The regular orbit where $n_i = 1$ for all $i \geq 1$ is a special case)

— Type C : $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$; $n_{1/2} = n_{3/2} + 1$, $n_\ell = 1, n_{\ell-1} = 2$ (The regular orbit where $n_i = 1$ for all $i \geq 1/2$ is a special case)

— Type D :

1. $n_i = 1$ for all $i \geq \ell$ and $n_0 = 1$, $n_i = 2$ for all $i \in \{2, \ldots, \ell - 1\}$.
2. $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 \geq 2$, $n_0 = \left\{ \begin{array}{l} \frac{n_1}{2} \text{ if } n_1 \text{ is even} \\ \frac{n_1+1}{2} \text{ if } n_1 \text{ is odd} \end{array} \right\}$

It will be denoted $(\underline{n})$.

The residual segment of type A (we say *linear residual segment*, refering to the general *linear* group) is characterized with the same three objects, and also corresponds bijectively to a weighted Dynkin diagram of type A. Then it is a decreasing sequence of (not necessarily positive) reals and the infinite tuple given above is $(\ldots, 0, 1, 1, 1, \ldots, 1)$, i.e $n_i \leq 1$ for all i. It is symmetrical around zero.

We will also abusively say *linear residual segment* for the translated version of a residual segment of type A ; i.e if it is not symmetrical around zero.

We usually do not write the commas to separate the (half)-integers in the sequence.

The use of the terminology « segments » is explained through the following example.

An example : Bernstein-Zelevinsky's segments

Consider the weighted Dynkin diagram of type A :

As $\langle \nu, \alpha_i \rangle = 1$ for all $i \iff \nu_i - \nu_{i+1} = 1$ for all i ; the vector of coordinates is therefore a strictly decreasing sequence of real numbers :$(a, a-1, a-2, \ldots, b)$.

The group GL_n is an example of reductive group whose root system is of type A.

We may now recall the notions of segments for GL_n as defined in Bernstein et Zelevinsky 1977, and following the treatment in Rodier 1981-1982. We fix an irreducible cuspidal representation ρ, and denote $\rho(a) = \rho|\text{det}|^a$. The representation $\rho_1 \times \rho_2$ denotes the parabolically induced representation from $\rho_1 \otimes \rho_2$.

Definition 26 (Segment, Linked segments)**.** [Bernstein-Zelevinsky ; following Rodier 1981-1982] A segment is an isomorphism class of irreducible cuspidal representations of a group GL_n, of the form $\mathcal{S} = \{\rho, \rho(1), \rho(2), \ldots, \rho(r-1)\}$. We denote it $\mathcal{S} = [\rho, \rho(r-1)]$.

There is also a notion of intersection and union of two such segments explained in particular in Rodier 1981-1982 : the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is written $\mathcal{S}_1 \cap \mathcal{S}_2$, the union is written $\mathcal{S}_1 \cup \mathcal{S}_2$.

Let $\mathcal{S}_1 = [\rho_1, \rho_1'], \mathcal{S}_2 = [\rho_2, \rho_2']$ be two segments. We say $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked if $\mathcal{S}_1 \not\subseteq \mathcal{S}_2, \mathcal{S}_2 \not\subseteq \mathcal{S}_1$ and $\mathcal{S}_1 \cup \mathcal{S}_2$ is a segment.

Once ρ is fixed, a segment is solely characterized by a string of (half)-integers, it seems therefore natural, in analogy with Bernstein- Zelevinsky's theory, to name any vector $(\nu_1, \ldots, \nu_k)$ corresponding to a dominant residual point and therefore by Proposition 24 (see also 102 and 103) to a weighted Dynkin diagram : *a residual segment*.

If $\mathcal{S} = [\rho, \rho(r-1)]$ is a segment, the unique irreducible subrepresentation of $\rho \times \ldots \times \rho(r-1)$ is denoted $Z(\mathcal{S})$.

Further, it is well-known that $Z(\mathcal{S})$ is the unique essentially square-integrable subrepresentation in the induced module $\rho \times \ldots \times \rho(r-1)$. Often, we denote it $Z(\rho, r-1, 0)$, and more generally $Z(\rho, \mathfrak{a}, \mathfrak{b})$ for $\mathfrak{a}$ and $\mathfrak{b}$ any two real numbers such that $a - b \in \mathbb{Z}$. In the literature, the generalized Steinberg is also denoted $\text{St}_k(\varrho)$, it is the canonical discrete series associated to the segment $[\varrho(\frac{k-1}{2}), \ldots, \varrho(\frac{1-k}{2})]$, for an irreducible cuspidal representation ϱ. Often, $\text{St}_k(\mathbf{1})$ will simply be denoted St_k.

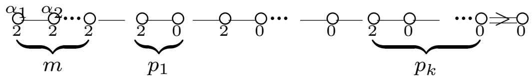
This is a general phenomenon, since by Theorem 11, for any quasi-split reductive group, we associate to any residual segment an essentially square- integrable (resp. discrete series) representation.

The well-known example of the Steinberg representation of GL_k is also characteristic since the Steinberg is the unique irreducible *generic* subquotient in the parabolically induced representation $\varrho(\frac{1-k}{2}) \times \ldots \times \varrho(\frac{1-k}{2})$.

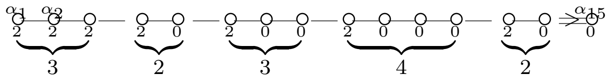
By Theorems 11 and 33, combined with Rodier's result, if the cuspidal support σ_λ, a residual point, is generic, then the induced representation is generic and the unique irreducible generic subquotient is essentially square integrable.

Therefore, the phenomenon presented here with the Steinberg subquotient, occurs more generally. When the generic representation σ_λ is a dominant residual point, the residual segment corresponding to λ characterizes the unique irreducible generic discrete series (resp. essentially square integrable) subquotient.

Example 3. Consider this example of type B :



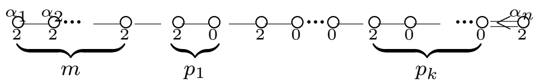
Consider B_{15} for instance, with $m = 3, p_1 = 2, p_2 = 3, p_3 = 4, p_4 = 2$:



We have $\langle \nu, \alpha_{15} \rangle = \langle \nu, 2e_{15} \rangle = 0$ and therefore $\nu_{15} = 0$.
$\langle \nu, \alpha_{14} \rangle = 0$ and therefore $\nu_{14} = \nu_{15} = 0$; $\langle \nu, \alpha_{13} \rangle = 1$, so $\nu_{13} - \nu_{14} = 1$.
Eventually the vector of coordinates corresponding to a dominant residual point, ν is

$$(\nu_1, \nu_2, \nu_3, \ldots, \nu_{13}, \nu_{14}, \nu_{15}) = (765433222111100)$$

Example 4. From the weigthed Dynkin diagram of type C_n :



We observe that $\langle \nu, \alpha_n \rangle = \langle \nu, 2e_n \rangle = 1$ and therefore $\nu_n = 1/2$ for some $i \leq n-1$, the weight under α_i is 2 and $\nu_i = 3/2$, etc. Residual segments of type C are therefore composed of half-integers.

1.5.2. Set of Jumps associated to a residual segment

In a following subsection (1.8.1), we will present certain embeddings of generic discrete series in parabolically induced modules. The proof of these embeddings necessitates to introduce the definition of the *set of Jumps* associated to a residual segment and therefore, transitively, to an irreducible generic discrete series.

These *Jumps* compose a finite set, *set of Jumps*, of (half)-integers a_i's, such that the set of integers $2a_i + 1$ is of a given parity. In the context of classical groups, the latter set (composed of elements of a given parity) coincides with the *Jordan block* defined in Moeglin et Tadic 2002. We will also use the notion of Jordan block in this subsection.

Let us recall our steps so far.

If we are given π_0, an irreducible generic discrete series of G, by Proposition 12 and Theorem 11, it embeds as a subrepresentation in $I_P^G(\sigma'_{\lambda'})$ for $\sigma'_{\lambda'}$ a dominant residual point. Further, by the results of Heiermann 2006 (see in particular Proposition 6.2), $\sigma'_{\lambda'}$ corresponds to a distinguished unipotent orbit and therefore a weigthed Dynkin diagram. Once $\Sigma_{\sigma'}$ is fixed (see the Subsection 1.5 or the introduction for the Definition of $\Sigma_{\sigma'}$), and assuming it is irreducible, the type of weighted Dynkin diagram is given. All details will be given in the next Section

1.5.3. By the previous argumentation (Subsection 1.5.1), we associate a residual segment $(\underline{n_{\pi_0}})$ to the irreducible generic discrete series π_0.

We illustrate these steps in the following example :

Example 5 (classical groups)**.** Let σ_λ be the cuspidal support of a generic discrete series π of a classical group (or its variants) $G(n)$, of rank n. First, assume $\sigma_\lambda := \rho|.|^a \otimes \ldots \rho|.|^b \otimes \sigma_c$ where ρ is a unitary cuspidal representation of GL_k, and σ_c a generic cuspidal representation of $G(k'), k' < n$. Using Bala-Carter theory, since λ is a residual point, it is in the Weyl group orbit of a dominant residual point, which corresponds to a weighted Dynkin diagram of type B (resp.C, D) and further the above sequence of exponents $(a, \ldots, b)$ is encoded $(\ell+m, \ldots, \ell, \ell-1, \ell-1, \ldots, 0) := (\underline{n})$ of type B (resp C, D). The type of weighted diagram only depends on the reducibility point of the induced representation of $G(k+k')$: $I^{G(k+k')}(\rho|.|^s \otimes \sigma_c)$ as explained in Proposition 24.

The bijective correspondence between *Residual segments* and *set of Jumps*

Let us start with the bijective map :

$(\underline{n}) \rightarrow$ set of Jumps of $(\underline{n})$

The length of a residual segment is the sum of the multiplicities : $n_{\ell+m} + n_{\ell+m-1} + \ldots n_1 + n_0$.

We first write a length d residual segment $(\underline{n})$

$$((\ell+m), \ldots, \underbrace{\ell}_{n_\ell \text{ times}} ; \underbrace{\ell-1}_{n_{\ell-1} \text{ times}}, \ldots, \underbrace{1}_{n_1 \text{ times}} \underbrace{0}_{n_0 \text{ times}})$$

as a length $2d+1$ (resp. $2d$) sequence of exponents (betokening an unramified character of the corresponding classical group, e.g. to B_d corresponds SO_{2d+1})

$$((\ell+m), \ldots, \underbrace{\ell}_{n_\ell \text{ times}} ; \underbrace{\ell-1}_{n_{\ell-1} \text{ times}}, \ldots, \underbrace{1}_{n_1 \text{ times}} \underbrace{0}_{n_0 \text{ times}}, 0,$$
$$\underbrace{0}_{n_0 \text{ times}} \underbrace{-1}_{n_1 \text{ times}} \ldots \underbrace{-\ell}_{n_\ell \text{ times}}, \ldots, -(\ell+m))$$

for type B_d only, we add the central zero

It is a decreasing sequence of $2d+1$ (for type B_d) or $2d$ (for type C_d, D_d) (half)-integers; from the previous Subsection (1.5.1), the reader has noticed that for $C_d, n_0 = 0$.

Then, we decompose this decreasing sequence as a multiset of $2n_0+1$ (resp. $2n_1$ for type D_d or $2n_{1/2}$ for type C_d) (it is the number of elements in the Jordan block) linear residual segments symmetrical around zero :

$$\{(a_1, a_1-1, \ldots, 0, \ldots, -a_1); (a_2, a_2-1, \ldots, 0, \ldots, -a_2); \ldots$$
$$\ldots; (a_{2n_0+1}, a_{2n_0+1}-1, \ldots, 0, \ldots, -a_{2n_0+1})\}$$

(resp.

$$\{(a_1, a_1-1, \ldots, 1/2, -1/2, \ldots, -a_1); (a_2, a_2-1, \ldots, 1/2, -1/2, \ldots, -a_2); \ldots$$
$$\ldots; (a_{2n_{1/2}}, a_{2n_{1/2}}-1, \ldots, 1/2, -1/2, \ldots, -a_{2n_{1/2}})\Big\}$$

where a_1 is the largest (half)-integer in the above decreasing sequence, a_2 is the largest (half)-integer with multiplicity 2, and in general a_i is the largest (half)-integer with multiplicity i.

Definition 27 (set of Jumps)**.** The *set of Jumps* is the set :

$$\{a_1, \ldots, a_{2n_0+1}\}$$

(resp. $\Big\{a_1, \ldots, a_{2n_{1/2}}\Big\}$). As one notices, the terminology comes from the observation that multiplicities at each jump increases by one : $n_{a_{i+1}} = n_{a_i} + 1$.

Let us make a parallel for the reader familiar with Moeglin-Tadic terminology for classical groups [Moeglin et Tadic 2002] (see also Tadic's notes « On classification of some classes of irreducible representations of classical groups » and « Reducibility and discrete series, in the case of classical p-adic groups ; an approach based on examples » for an introductory summary of these notions). In such context the Jordan block of the irreducible discrete series π associated to the residual segment $(\underline{n})$ (denoted Jord$_\pi$) is constituted of the integers :

$$\{2a_1+1, 2a_2+1; \ldots, 2a_{2n_0+1}+1\}$$

(resp. $\Big\{2a_1+1, 2a_2+1; \ldots, 2a_{2n_{1/2}}+1\Big\}$). This is not a complete characterization of a Jordan block : for a correct use of the definition of Jordan block, we should also fix a self-dual irreducible cuspidal representation ρ of a general linear group and an irreducible cuspidal representation σ_c of a smaller classical group.

We *abusively* use the terminology *Jordan block* to define one partition but such partition is only one of the constituents of the Jordan block as defined in Moeglin et Tadic 2002.

Clearly the Jordan block is a set of distinct odd (resp even) integers. According to Moeglin et Tadic 2002, the following condition should also be satisfied : $2d+1 = \sum_i (2a_i+1)$ for type B (resp. $2d = \sum_i (2a_i+1)$ for type C).

Moreover, we are now going to explain there is a canonical way to obtain for a given type $(A, B, C$, or $D)$ and a fixed length d all distinguished nilpotent orbits, thus all Weighted Dynkin diagrams and therefore all residual segments of these given type and length.

This is given by Bala-Carter theory (see the Appendix F and in particular the Theorem 101). First, one should partition the integer $2d+1$ (resp $2d$) into distinct odd (resp. even) integers (given $2d+1$, or $2d$ there is a finite number of such partitions). Each partition corresponds to a distinguished orbit and further to a dominant residual point, hence a residual segment.

In fact, each partition corresponds to a Jordan block of an irreducible discrete series π (whose associated residual segment is $(\underline{n_\pi})$). Let us detail the three cases (B, C and D).

Let us finally illustrate the following correspondence :

$$\text{Jord}_\pi \to \text{set of Jumps } (\underline{n_\pi}) \to (\underline{n_\pi})$$

— In case of B_d, the set Jumps of (n_π) derives easily from the choice of *one* partition of $2d+1$ in distinct odd integers : Jord$_\pi = \{2a_1 + 1, 2a_2 + 1, \dots, 2a_t + 1\}$. Then Jumps of $(n_\pi) = \{a_1, a_2, \dots, a_t\}$.

  Once this set of Jumps identified, one writes the corresponding symmetrical around zero linear segments $(a_i, \dots, -a_i)$'s and by combining and reordering them, form a decreasing sequence of integers of length $2d + 1$.

  This length $2d+1$ sequence is symmetrical around zero, with a length d sequence of positive elements, a central zero, and the symmetrical sequence of negative elements. The length d sequence of positive elements is the residual segment $(\underline{n})$.

— Again the case of C_d (by Theorem 101 in Appendix F) $2d$ is partitioned into distinct even integers, each partition corresponds to a distinguished orbit and further to a dominant residual point, hence a residual segment.

  The correspondence is the following : to the Jordan block of a generic discrete series, π and its associated residual segment $\underline{n_\pi}$:

  Jord$_\pi = \{2a_1 + 1, 2a_2 + 1, \dots, 2a_t + 1\}$, for each a_i, one writes $(a_i, a_i-1, \dots, 1/2, -1/2, \dots - a_i)$. One takes all elements in all these sequences, reorder them to get a $2d$ decreasing sequence of half-integers. The length d sequence of positive half-integers corresponds to residual segment $(\underline{n})$ of type C_d.

— In case of D_d, let Jord$_\pi = \{2a_1 + 1, 2a_2 + 1, \dots, 2a_t + 1\}$ be the Jordan block of a generic discrete series, π ; then write the corresponding linear segments $(a_i, \dots, -a_i)$'s, with all these residual segments, form a decreasing sequence of integers of length $2d$. This length $2d$ sequence is symmetrical around zero. The length d sequence of positive elements in chosen to form the residual segment $(\underline{n})$.

Example 6 (B_{14}). Let us consider one partition of 2.14+1 into distinct odd integers : $\{11, 9, 5, 3, 1\}$.

For each odd integer in this partition, write it as $2a_i + 1$ and write the corresponding linear residual segments $(a_i, \dots, -a_i)$:

$$543210-1\ \ -2\ \ -3\ \ -4\ \ -5$$

$$43210-1\ \ -2\ \ -3\ \ -4$$

$$210-1\ \ -2$$

$$10-1$$

$$0$$

Re-assembling, we get

$$54433222111100; 0; 0\ 0\ \ -1\ \ -1\ \ -1\ \ -1\ \ -2\ \ -2\ \ -2\ \ -3\ \ -3\ \ -4\ \ -4\ \ -5$$

Then, the corresponding residual segment of length 14 (29=2.14+1) is : 54433222111100.

Example 7 (C_9)**.** Then $2d'_i$ is 18, and we decompose 18 into distinct even integers : 18 ; 14+4 ; 12+4+2 ; 16+2 ; 8+6+4, 12+6, 10+8. To each of these partitions corresponds the Weyl group orbit of a residual point and therefore a residual segment. The regular orbit (since the exponents of the associated residual segment form a regular character of the torus) correspond to 18. It is simply

$$(17/2, 15/2, 13/2, \dots, 1/2)$$

The half-integer 17/2 is such that 2(17/2) + 1 =18.

Let us consider the third partition, 12+4+2, : 12= 2(11/2) + 1 ; 4= 2(3/2) + 1 ; 2 = 2(1/2) + 1. Each even integer gives a strictly decreasing sequence of half-integers (11/2,9/2,7/2,5/2,3/2,1/2) ; (3/2,1/2) ; (1/2). Finally, we reorder the nine half-integers obtained as a decreasing sequence :

$$(11/2, 9/2, 7/2, 5/2, 3/2, 3/2, 1/2, 1/2, 1/2)$$

The 6 other partitions correspond to :
$(15/2, 13/2, 11/2, 9/2, 7/2, 5/2, 3/2, 1/2, 1/2)$; $(11/2, 9/2, 7/2, 5/2, 5/2, 3/2, 3/2, 1/2, 1/2)$
$(13/2, 11/2, 9/2, 7/2, 5/2, 3/2, 3/2, 1/2, 1/2)$; $(9/2, 7/2, 7/2, 5/2, 5/2, 3/2, 3/2, 1/2, 1/2)$;
$(7/2, 5/2, 5/2, 3/2, 3/2, 3/2, 1/2, 1/2, 1/2)$

A few more examples of type B and D are treated in Appendix B.

Remark 4. Once given a residual segment, $(\underline{n})$, and its corresponding set of Jumps $a_1 > a_2 > \dots > a_n$, one observes that for any i, $(a_i, \dots, -a_{i+1})(\underline{n_i})$ is in W_σ-orbit of this residual segment, where $(a_i, \dots, -a_{i+1})$ is a linear residual segment and $(\underline{n_i})$ a residual segment of the same type as $(\underline{n})$.

Therefore a set of asymmetrical linear segments $(a_i, \dots, -a_{i+1})$ along with the smallest residual segment of a given type (e.g (100) for type B, resp. $(3/2, 1/2, 1/2)$ for type C) *or* a linear segments $(a_1, a_1-1, \dots 0)$ (resp. $(a_1, a_1-1, \dots 1/2)$ for type C) is in the W_σ-orbit of the residual segment $(\underline{n})$.

Clearly, a set of linear *symmetrical* segments cannot be in the W_σ-orbit of the residual segment $(\underline{n})$.

1.5.3. Application of the theory of residual segments : reformulation of our setting

1.5.3.1. Reformulation of our setting

Let us come back to our setting (recalled at the beginning of the Subsection 1.5).

Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1. Put $\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{M_{1,\alpha}}(\sigma) = 0\}$ (resp. $\Sigma_\sigma^M = \{\alpha \in \Sigma_{red}^M(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$). The set Σ_σ is a root system in a subspace of $a_{M_1}^G *$ (resp. $(a_{M_1}^M)^*$)(cf. Silberger 1981 3.5).

Suppose that the irreducible components of Σ_σ are all of type A, B, C or D.

First assume Σ_σ is irreducible and let us denote $\mathcal{T}$ its type, and $\Delta_\sigma := \{\alpha_1, \ldots, \alpha_d\}$ the basis of Σ_σ (following our choice of basis for the root system of G).

We will consider maximal standard Levi subgroups of G, $M \supset M_1$, corresponding to sets $\Delta - \{\overline{\alpha_k}\}$, for a simple root $\overline{\alpha_k} \in \Delta$ (here we use the notation $\overline{\alpha_k}$ to avoid confusion with the roots in Δ_σ).

If $\overline{\alpha_k}$ is not a extremal root of the Dynkin diagram of G, Σ^M decomposes in two disjoints components.

Then, Σ_σ^M is a disjoint union of two irreducible components $\Sigma_{\sigma,1}^M \bigcup \Sigma_{\sigma,2}^M$ of type A and $\mathcal{T}$, one of which may be empty (if we remove extremal roots from the Dynkin diagram).

If we remove $\overline{\alpha_n}$, $\Sigma_{\sigma,2}^M$ is empty, and $\Sigma_{\sigma,1}^M$ is of type A, whereas if we remove $\overline{\alpha_1}$, $\Sigma_{\sigma,2}^M$ is of type $\mathcal{T}$ and $\Sigma_{\sigma,1}^M$ is empty.

Else we assume Σ_σ is not irreducible but a disjoint union of irreducible components or empty components $\Sigma_{\sigma,i}$ for $i = 1, \ldots, r$ of type A, B, C or D : $\Sigma_\sigma = \bigcup_i \Sigma_{\sigma,i}$.

Then, the basis of Σ_σ is

$$\Delta_\sigma := \{\alpha_{1,1}, \ldots, \alpha_{1,d_1}; \alpha_{2,1}, \ldots, \alpha_{2,d_2}, \ldots, \alpha_{i,1}, \ldots, \alpha_{i,d_i}, \ldots, \alpha_{r,1}, \ldots, \alpha_{r,d_r}\}$$

Again, we will consider maximal standard Levi subgroup of G, $M \supset M_1$, corresponding to sets $\Delta - \{\overline{\alpha_k}\}$.

Then, for an index $j \in \{1, \ldots, r\}$, $\Sigma_{\sigma,j}^M$ is a disjoint union of two irreducible components $\Sigma_{\sigma,j_1}^M \bigcup \Sigma_{\sigma,j_2}^M$ of type A and $\mathcal{T}$, one of which may be empty (if $\overline{\alpha_k}$ is an « extremal » root of the Dynkin diagram of G).

If we remove the last simple root, $\overline{\alpha_n}$, of the Dynkin diagram, Σ_{σ,j_2}^M is empty, and Σ_{σ,j_1}^M is of type A, whereas if we remove α_1, Σ_{σ,j_2}^M is of type $\mathcal{T}$ and Σ_{σ,j_1}^M is empty.

Therefore, it will be enough to prove our results and statements in the case of Σ_σ irreducible ; since in case of reducibility, without loss of generality, we choose a component $\Sigma_{\sigma,j}$ and the same reasonings apply.

Now, in our setting (see the beginning of the Subsection 1.5), σ_ν is a residual point for μ^M. Recall Σ_σ is of rank $d = d_1 + d_2$. Therefore the residual point is in the cuspidal support of the generic discrete series τ if and only if (applying Proposition 24 above) : $rk(\Sigma_\sigma^M) = d_1 - 1 + d_2$.

We write $\Sigma_\sigma^M := A_{d_1-1} \bigcup \mathcal{T}_{d_2}$ and ν corresponds to residual segments $(\nu_{1,1}, \ldots, \nu_{1,d_1})$ and $(\nu_{2,1}, \ldots, \nu_{2,d_2})$.

Let us assume that the representation σ_λ is in the cuspidal support of the essentially square integrable representation of M, $\tau_{s\tilde{\alpha}}$, where $\lambda = \nu + s\tilde{\alpha}$. We add the twist $s\tilde{\alpha}$ on the linear part (i.e corresponding to A_{d_1-1}), and therefore $(\nu_{2,1}, \ldots, \nu_{2,d_2})$ is left unchanged and is thus $(\lambda_{2,1}, \ldots, \lambda_{2,d_2})$, whereas $(\nu_{1,1}, \ldots, \nu_{1,d_1})$ becomes $(\lambda_{1,1}, \ldots, \lambda_{1,d_1})$.

Then, we need to obtain from $(\lambda_{1,1}, \ldots, \lambda_{1,d_1})(\lambda_{2,1}, \ldots, \lambda_{2,d_2})$ a residual segment of length d and type $\mathcal{T}$.

Indeed, it is the only option to insure σ_λ is a residual point (applying Proposition 24) for μ^G, in particular, since $d = d_1 + d_2$ (and therefore writing $\Sigma_\sigma = A_{d_1-1} \bigcup \mathcal{T}_{d_2}$ does not satisfy the requirement of Proposition 24).

1.5.3.2. Cuspidal strings

Assume we remove a non-extremal simple root of the Dynkin diagram, the parameter λ in the cuspidal support is therefore constituted of a couple of residual segments, one of which is a linear residual segment : $(a, \ldots, b)$, and the other is denoted $(\underline{n})$. It will be convenient to define the cuspidal support to be given by the tuple $(a, b, \underline{n})$ where $\underline{n}$ is a tuple $(\ldots, 0, n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_1, n_0)$ characterization uniquely the residual segment. We define :

Definition 28 (cuspidal string)**.** Given two residual segments, strings of integers (or half-integers) : $(a, \ldots, b)(\underline{n})$. The tuple $(a, b, \underline{n})$ where $\underline{n}$ is the $(\ell + m + 1)$-tuple

$$(n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_1, n_0)$$

is named a cuspidal string.

Recall W_σ is the Weyl group of the root system Σ_σ.

Definition 29 (W_σ-cuspidal string)**.** Given a tuple $(a, b, \underline{n})$ where $\underline{n}$ is the $(\ell+m+1)$-tuple $(n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_1, n_0)$, the set of all tuples $(a', b', \underline{n'})$ where $\underline{n'}$ is a $(\ell' + m' + 1)$-tuple $(n'_{\ell'+m'}, \ldots, n'_{\ell'}, n'_{\ell'-1}, \ldots, n'_1, n'_0)$ in the W_σ orbit of $(a, b, \underline{n})$ is called W_σ-cuspidal string.

Remark 5. These definitions can be extended to include the case of t linear residual segments (i.e of type A) :$(a_1, \ldots, b_1)(a_2, \ldots, b_2) \ldots (a_t, \ldots, b_t)$ and a residual segment $(\underline{n})$ of type B, C or D, then the parameter in the cuspidal support will be denoted $(a_1, b_1; a_2, b_2; \ldots; a_t, b_t, \underline{n})$.

1.5.4. Application to the case of classical groups

We illustrate in the following subsection how these definitions naturally appear in the context of classical groups.

1.5.4.1. Unramified principal series

Let τ be a generic discrete series of $M = M_L \times M_c$, the maximal Levi subgroup in a classical group G, $M_L \subset P_L$ is a linear group and $M_c \subset P_c$ is a smaller classical group. It is a tensor product of an essentially square integrable representation of a linear group and an irreducible generic discrete series π of a smaller classical group of the same type as G.

$$\tau := St_{d_1}|.|^s \otimes \pi, \text{ with } s = \frac{\mathfrak{a}+\mathfrak{b}}{2}$$

Further, let us assume $(P_1, \sigma, \lambda) := (P_0, \mathbf{1}, \lambda)$. The twisted Steinberg is the unique subrepresentation in $I_{P_{0,L}}^{M_L}(\mathfrak{a}, \ldots, \mathfrak{b})$, whereas $\pi \hookrightarrow I_{P_{0,c}}^{M_c}(\underline{n})$.

Therefore,

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_c \times P_L}^G(I_{P_{0,L}}^{M_L}(\mathfrak{a}, \ldots, \mathfrak{b}) I_{P_{0,c}}^{M_c}(\underline{n})) \cong I_{P_0}^G((\mathfrak{a}, \ldots, \mathfrak{b})(\underline{n}))$$

1.5.4.2. The general case

Assume τ is an irreducible generic essentially square integrable representation of a maximal Levi subgroup M of a classical group of rank $\sum_{i=1}^r d_i.\dim(\sigma_i)+k$. Then $\tau := St_{d_1}(\sigma_1)|.|^s \otimes \pi$, with $s = \frac{\mathfrak{a}+\mathfrak{b}}{2}$.

We study the cuspidal support of the generic (essentially) square integrable representations $St_{d_1}(\sigma_1)|.|^s$ and π.

By Proposition 12, $\pi \hookrightarrow I_{P_{1,c}}^{M_c}(\sigma_{\nu_c}^c)$ such that :

$$M_{1,c} = \underbrace{GL_{k_2} \times \ldots \times GL_{k_2}}_{d_2 \text{ times}} \times \ldots \times \underbrace{GL_{k_r} \times \ldots \times GL_{k_r}}_{d_r \text{ times}} \times G(k)$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G.

We write the cuspidal representation $\sigma^c := \sigma_2 \otimes \ldots \sigma_2 \otimes \ldots \otimes \sigma_r \otimes \ldots \sigma_r \otimes \sigma_c$ of $M_{1,c}$ and assume the inertial classes of the representations of GL_{k_i}, σ_i, are mutually distinct and $\sigma_i \cong \sigma_i^\vee$ if $\sigma_i, \sigma_i^\vee$ are in the same inertial orbit.

The residual point ν_c is dominant : $\nu_c \in ((a_{M_1}^M)^*+$. Applying Proposition 24 below with ν_c and the root system Σ_σ^M, we have :

$$\nu_c := (\nu_2, \ldots, \nu_r)$$

where each ν_i for $i \in \{2, \ldots, r\}$ is a residual point, corresponding to a residual segment of type $B_{d_i}, C_{d_i}, D_{d_i}$.

Further,

$$\mathrm{St}_{d_1}(\sigma_1)|.|^s \hookrightarrow I_{P_{1,L}}^{M_L}(\sigma_1, \lambda_L) \cong I_{P_{1,L}}^{M_L}(\sigma_1|.|^{a} \otimes \sigma_1|.|^{a-1} \dots \sigma_1|.|^{b})$$

where λ_L is the residual segment of type A : $(a, a-1, \dots, b)$, and M_L is the linear part of Levi subgroup M.

Such that eventually :

$$\sigma = \sigma_1 \otimes \sigma_1 \dots \sigma_1 \otimes \sigma_2 \otimes \dots \sigma_2 \otimes \dots \otimes \sigma_r \otimes \dots \sigma_r \otimes \sigma_c$$

And σ_λ can be rewritten :

$$\sigma_1|.|^{a} \otimes \sigma_1|.|^{a-1} \dots \sigma_1|.|^{b} \otimes \underbrace{\sigma_2|.|^{\ell_2} \dots \sigma_2|.|^{\ell_2}}_{n_{\ell_2} \text{ times}} \dots \underbrace{\sigma_2|.|^{0} \dots \otimes \sigma_2|.|^{0}}_{n_{0,2} \text{ times}} \dots$$
$$\underbrace{\sigma_r|.|^{\ell_r} \dots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \dots \underbrace{\otimes \sigma_r|.|^{0} \dots \otimes \sigma_r|.|^{0}}_{n_{0,r} \text{ times}} \otimes \sigma_c \quad (1.5)$$

The character ν, representation of M_1, can be splitted in two parts ν_1 and $\underline{\nu} = (\nu_2, \dots, \nu_r)$, residual points, giving the discrete series denoted $\mathrm{St}_{d_1}(\sigma_1)$ in $I_{P_{1,L}}^{M_L}(\sigma_1)$ and π in $I_{P_{1,c}}^{M_c}(\sigma_c, \underline{\nu})$. By a simple computation, it can be shown that the twist $s\tilde{\alpha}$ will be added on the 'linear part' of the representation and leaves the semi-simple part (classical part) invariant.

Namely ν is given by a vector $(\nu_1 = 0, \nu_2, \dots, \nu_r)$ and we add the twist $s\tilde{\alpha}$ on the first element to get the vector : $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ where each λ_i is a residual segment $(\underline{n_i})$ associated to the subsystem $\Sigma_{\sigma,i}$.

To use the bijection between W_σ orbits of residual points and weighted Dynkin diagrams, one needs to use a certain root system and its associated Weyl group. Then λ is a tuple of r residual segments of different types : $\{(\underline{n_i})\}, i \in \{1, \dots, r\}$. If the parameter λ is written as a r-tuple : $(\lambda_1, \dots, \lambda_r)$, it is dominant if and only if each λ_i is dominant with respect to the subsystem $\Sigma_{\sigma,i}$.

We have not yet used the *genericity* property of the cuspidal support. This is where we use Proposition 24. The generic representation σ_c and the reducibility point of the representation induced from $\sigma_i|.|^s \otimes \sigma_c$ determine the type of the residual segment $(\underline{n_i})$ obtained.

1.6. Characterization of the unique irreducible generic subquotient in the standard module

1.6.1.

Let us first outline the results presented in this section.

Let us assume that the irreducible generic subquotient in the standard module is not discrete series. We characterize the Langlands parameter of this unique irreducible non-square integrable subquotient using an order on Langlands parameters given in Lemma 31 below : more precisely, in Theorem 35, we prove this unique irreducible generic subquotient is identified by its Langlands parameter being minimal for this order.

We compare Langlands parameters in the Subsection 1.6.2, and along those results and Theorem 35, we will prove a lemma (Lemma 42) in the vein of Zelevinsky's Theorem (in the following Section 1.7).

Finally, before entering the next section where we prove the Conjecture for Σ_σ of type A, we need to come back on the depiction of the intertwining operators used in our context. This subsection 1.6.3 on intertwining operators also contains a lemma (Lemma 39) which is crucial in the proof of the Conjecture.

Using Langlands' classification (see Theorem 13) and the *Standard module conjecture* (see Theorem 14), we can characterize the unique irreducible generic non-square integrable subquotient, denoted $I_{P'}^G(\tau'_{\nu'})$. In particular, on a given cuspidal support, we can characterize the form of the Langlands' parameter ν'. We introduce the necessary tools and results regarding this theory in this subsection.

To study subquotients in the standard module induced from a maximal parabolic subgroup P, $I_P^G(\tau_{s\tilde{\alpha}})$, we will use the following well-known lemma from Borel et Wallach 1999 :

Let us recall their definition of the order :

Definition 30 (order)**.** $\lambda_\mu \leq \lambda_\pi$ if $\lambda_\pi - \lambda_\mu = \sum_i x_i\alpha_i$ for simple roots α_i in a_0^* and $x_i \geq 0$.

Lemma 31 (Borel-Wallach, 2.13 in Chapter XI of Borel et Wallach 1999)**.** *Let P, σ, λ_π be Langlands data. If (μ, H) is a constituent of $I_P^G(\sigma_{\lambda_\pi})$ the standard module, and if $\pi = J(P, \sigma, \lambda_\pi)$ is the Langlands quotient, then $\lambda_\mu \leq \lambda_\pi$, and equality occurs if and only if μ is $J(P, \sigma, \lambda_\pi)$.*

We will write this order on Langlands parameters :

$$\lambda_\mu {}_P\!\leq \lambda_\pi$$

Lemma 32. *Let $\nu = \sum_{i=1}^n a_i e_i$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$. $0_P\!\leq \nu$ if and only if $\sum_{i=1}^k a_i \geq 0$ for any k in non-D_n cases. In the case of D_n, one needs to specify $\sum_{i=1}^k a_i \geq 0$ for any $k \leq n-1$, $a_{n-1} \geq -a_n$ and $a_{n-1} \geq a_n$.*

Proof. From the expression $\nu = \sum_{i=1}^n a_i e_i$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$, we can recover an expression of ν in the canonical basis of the Lie algebra $a_0^* : \nu = \sum_{i=1}^n x_i \alpha_i$. Let's explicit $\nu = \sum_i x_i \alpha_i$:

$$\nu = \sum_{i=1}^{n-1} x_i(e_i - e_{i+1}) + x_n \alpha_n =$$

$$x_1(e_1-e_2)+x_2(e_2-e_3)+x_3(e_3-e_4)+\ldots+x_{n-1}(e_{n-1}-e_n)\begin{cases} & \text{for } A_{n-1} \\ +x_n(e_{n-1}+e_n) & \text{for } D_n \\ +x_n e_n & \text{for } B_n \\ +2x_n e_n & \text{for } C_n \end{cases}$$

Then,

$$\nu = \sum_{i=1}^n a_i e_i = x_1 e_1+(x_2-x_1)e_2+(x_3-x_2)e_3+\ldots+\begin{cases} (x_{n-1}-x_{n-2})e_{n-1} - x_{n-1}e_n & \text{for } A_{n-1} \\ (x_{n-1}+x_n)e_{n-1} + (x_n - x_{n-1})e_n & \text{for } D_n \\ (x_{n-1}-x_{n-2})e_{n-1} + (x_n - x_{n-1})e_n & \text{for } B_n \\ (x_{n-1}-x_{n-2})e_{n-1} + (2x_n - x_{n-1})e_n & \text{for } C_n \end{cases}$$

$\nu = \sum_{i=1}^n x_i \alpha_i \geq 0 \Leftrightarrow x_i \geq 0 \ \forall i$

From above $x_1 = a_1, x_2 - x_1 = a_2 \Leftrightarrow x_2 = a_1 + a_2, \ldots$ We have : $x_k = \sum_{i=1}^k a_i \ \forall k$ except for root system of type D_n, where for index $n-1$ and n, $2x_n = \sum_{i=1}^{n-1} a_i + a_n$ and $2x_{n-1} = \sum_{i=1}^{n-1} a_i - a_n$, and for C_n where $2x_n = \sum_{i=1}^n a_i$.

Notice that for A_{n-1}, $x_{n-1} = \sum_{i=1}^{n-1} a_i$ and $a_n = -x_{n-1}$ such that $\sum_{i=1}^n a_i = 0$.

Therefore $0_P \leq \nu$ if and only if $\sum_{i=1}^k a_i \geq 0$ for any k in non-D_n cases. In the case of D_n, one needs to specify $\sum_{i=1}^k a_i \geq 0$ for any $k \leq n-1$, $\sum_{i=1}^{n-1} a_i \geq -a_n$ and $\sum_{i=1}^{n-1} a_i \geq a_n$. □

The next result will be used in the course of the proof of the Generalized Injectivity Conjecture for non-discrete series subquotients presented in the Sections 1.9.1 and D.4.3. We use the notations of Section 1.3. We will need the following theorem :

Theorem 33. *[Theorem 2.2 of* Heiermann *et* Muic *2006]*

Let $P = MU$ be a F-standard parabolic subgroup of G and σ an irreducible generic cuspidal representation of M. If the induced representation $I_P^G(\sigma)$ has a subquotient which lies in the discrete series of G (resp. is tempered) then the unique irreducible generic sub-quotient of $I_P^G(\sigma)$ lies in the discrete series of G (resp. is tempered).

Lemma 34. *Let σ be an irreducible generic cuspidal representation of M_1 and σ_λ be a dominant residual point and consider the generic induced module $I_{P_1}^G(\sigma_\lambda)$. Its unique irreducible generic square-integrable subquotient is a subrepresentation.*

Proof. From Theorem 11, since λ is a residual point, $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient. From Rodier's Theorem, it also has a unique irreducible generic subquotient, denote it γ. From Theorem 33, this unique irreducible generic subquotient is discrete series. Consider this unique generic discrete series subquotient, by Proposition 12, there exists a parabolic subgroup P' such that $\gamma \hookrightarrow I_{P'}^G(\sigma'_{\lambda'})$, and λ' dominant for P'. Then the lemma follows from Proposition 16 in Section 1.4. $\square$

Theorem 35. *Let $I_P^G(\tau_\nu)$ be a generic standard module and (P', τ', ν') the Langlands data of its unique irreducible generic subquotient.*

If (P'', τ'', ν'') is the Langlands data of any other irreducible subquotient, then $\nu' {}_P\!\leq \nu''$. The inequality is strict if the standard module $I_{P''}^G(\tau''_{\nu''})$ is generic.

In other words, ν' is the smallest Langlands parameter for the order (defined in Lemma 31) among the Langlands parameters of standard modules having (σ, λ) as cuspidal support.

Proof. First using the result of Heiermann-Opdam (in Heiermann et Opdam 2009), we let $I_P^G(\tau_\nu)$ be embedded in $I_{P_1}^G(\sigma_{\nu_0+\nu})$ with cuspidal support $(\sigma, \lambda = \nu_0 + \nu)$.

Using Langlands' classification, we write $J(P', \tau', \nu')$ an irreducible generic subquotient of $I_P^G(\tau_\nu)$. Then the standard module conjecture claims that $J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'})$.

The first case to consider is a *generic* standard module $I_{P''}^G(\tau''_{\nu'})$. From the unicity of the generic irreducible module with cuspidal support (σ, λ) (Rodier's Theorem), one sees that $J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'}) \leq I_{P''}^G(\tau''_{\nu''})$.

Hence, $\nu' {}_{P''}\!< \nu''$.

Secondly, if the standard module $I_{P''}^G(\tau''_{\nu''})$ is any (non-generic) subquotient having (σ, λ) as cuspidal support, since this cuspidal support is generic one will see that one can replace τ'' by the generic tempered representation τ''_{gen} with same cuspidal support and conserve the Langlands parameter ν'' and we are back to the first case. This is explained in the next paragrapher. The lemma follows.

To replace the tempered representation τ'' of M'' the argument goes as follows : Since the representation σ in the cuspidal support of this representation is generic, by Theorem 33 the unique irreducible generic representation subquotient τ''_{gen} in the representation induced from this cuspidal support is tempered. As any representation in the cuspidal support of τ'' must lie in the cuspidal support of τ''_{gen}, any such representation must be conjugated to σ. That is there exists a Weyl group element $w \in W$ such that if

$$\tau'' \hookrightarrow I_{P_1 \cap M''}^{M''}(\sigma_{\nu_0})$$

then

$$\tau''_{\text{gen}} \hookrightarrow I_{P_1 \cap M''}^{M''}((w\sigma)_{w\nu_0})$$

Twisting by $\nu'' \in a_{M'}^*$ comes second. Therefore conjugation by this Weyl group element leaves invariant the Langlands parameter $\nu'' \in a_{M'}^*$, and $(\tau''_{\text{gen}})_{\nu''}$ and $\tau''_{\nu''}$ share therefore the same cuspidal support. $\square$

1.6.2. Linear residual segments

Let $I_P^G(\tau_{s\tilde{\alpha}})$ be a standard module, we call the parameter $s\tilde{\alpha}$ *the Langlands parameter of the standard module*. We have seen that this Langlands parameter (the *twist*) depends only on the linear (not semi-simple) part of the cuspidal support, i.e the linear residual segment.

In this section and the following we use the notation $\mathcal{S}$ (see the Definition 26) to denote a *linear residual segment*, the underlying irreducible cuspidal representation ρ is implicit.

A simple computation gives that if a standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where P is a maximal parabolic, embeds in $I_{P_1}^G(\sigma(a, b, \underline{n}))$ for a cuspidal string $(a, b, \underline{n})$, then $s = \frac{a+b}{2}$. The parameter $s\tilde{\alpha}$ is in $(a_M^*)^+$, but to use Lemma 31 we will need to consider it as an element of $a_{M_1}^*$.

Then, we say this Langlands parameter is *associated* to the linear residual segment $(a, \ldots, b)$. In this subsection, we compare Langlands parameters associated to linear residual segments.

Lemma 36. *Let γ be a real number such that $a \geq \gamma \geq b$.*

Splitting a linear residual segment $(a, \ldots, b)$ whose associated Langlands parameter is $\lambda = \frac{a+b}{2} \in a_M^*$ into two segments : $(a, \ldots, \gamma+1)(\gamma, b)$ yields necessarily a larger Langlands parameter, λ' for the order given in Lemma 31.

Proof. We write $\lambda \in a_M^*$ as an element in $a_{M_1}^*$ to be able to use Lemma 31 (i.e the Lemma 31 also applies with $a_{M_1}^*$) :

$$\lambda = (\underbrace{\frac{a+b}{2}, \ldots, \frac{a+b}{2}}_{a-b+1 \text{ times}})$$

Similarly, we write λ'

$$\lambda' = (\underbrace{\frac{a+(\gamma+1)}{2}, \ldots, \frac{a+(\gamma+1)}{2}}_{a-\gamma \text{ times}}, \underbrace{\frac{\gamma+b}{2}, \ldots, \frac{\gamma+b}{2}}_{\gamma-b+1 \text{ times}})$$

$$\lambda' - \lambda = (\underbrace{\frac{(\gamma+1)-b}{2}, \ldots, \frac{(\gamma+1)-b}{2}}_{a-\gamma \text{ times}}, \underbrace{\frac{\gamma-a}{2}, \ldots, \frac{\gamma-a}{2}}_{\gamma-b+1 \text{ times}})$$

Therefore, $x_1 = \frac{(\gamma+1)-b}{2} > 0$. Since $x_k = \sum_{i=1}^k a_i$ as written in the proof of Lemma 32, one observes that $x_k > x_n$ for any $k < n = a - b + 1$, and $x_n = \frac{(\gamma+1)-b}{2}(a-\gamma) + \frac{\gamma-a}{2}(\gamma - b + 1) = (a-\gamma)(\frac{(\gamma+1)-b}{2} - \frac{-\gamma+b-1}{2}) = 0$. Hence $\lambda' \geq_P \lambda$ by Lemma 32. □

Proposition 37. *Consider two linear (i.e of type A) residual segments, i.e stricly decreasing sequences of real numbers such that the difference between two consecutive* *reals is one :* $\mathcal{S}_1 := (a_1, \dots, b_1); \mathcal{S}_2 := (a_2, \dots, b_2)$*. Typically, one could think of decreasing sequences of consecutive integers or consecutive half-integers.*

Assume $a_1 > a_2 > b_1 > b_2$ *so that they are linked in the terminology of Bernstein-Zelevinsky. Taking intersection and union yield two unlinked residual segments* $\mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathcal{S}_1 \cup \mathcal{S}_2$*.*

Denote $\lambda \in a_M^*$ *the Langlands parameter* $\lambda = (s_1, s_2)$ *associated to* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, and expressed in the canonical basis associated to the Lie algebra* a_0^**.*

Denote $\lambda' \in a_M^* : \lambda' = (s_1', s_2')$ *the one associated to the two unlinked segments* $\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \cup \mathcal{S}_2$ *ordered so that* $s_1' > s_2'$*.*

Then, $\lambda' {}_P\!\leq \lambda$*.*

Proof. Let $(a_1, \dots, b_1)(a_2, \dots, b_2)$ be two segments with $a_1 > a_2 > b_1 > b_2$ so that the two segments are linked. The associated Langlands parameter is :

$$\lambda = (\underbrace{\frac{a_1+b_1}{2}, \dots, \frac{a_1+b_1}{2}}_{a_1-b_1+1 \text{ times}}, \underbrace{\frac{a_2+b_2}{2}, \dots, \frac{a_2+b_2}{2}}_{a_2-b_2+1 \text{ times}})$$

Then taking union and intersection of those two segments gives : $(a_1, \dots, b_2)(a_2, \dots, b_1)$ or $(a_2, \dots, b_1)(a_1, \dots, b_2)$ ordered so that $s_1' > s_2'$.

The Langlands parameter will therefore be given by :

1. If $\frac{a_1+b_2}{2} \geq \frac{a_2+b_1}{2}$:

$$\lambda' = (\underbrace{\frac{a_1+b_2}{2}, \dots, \frac{a_1+b_2}{2}}_{a_1-b_2+1 \text{ times}}, \underbrace{\frac{a_2+b_1}{2}, \dots, \frac{a_2+b_1}{2}}_{a_2-b_1+1 \text{ times}})$$

2. If $\frac{a_2+b_1}{2} > \frac{a_1+b_2}{2}$:

$$\lambda' = (\underbrace{\frac{a_2+b_1}{2}, \dots, \frac{a_2+b_1}{2}}_{a_2-b_1+1 \text{ times}}, \underbrace{\frac{a_1+b_2}{2}, \dots, \frac{a_1+b_2}{2}}_{a_1-b_2+1 \text{ times}})$$

Then the difference $\lambda - \lambda'$ equals :

— In case (1)

$$(\underbrace{\frac{b_1-b_2}{2}, \dots, \frac{b_1-b_2}{2}}_{a_1-b_1+1 \text{ times}}, \underbrace{\frac{a_2-a_1}{2}, \dots, \frac{a_2-a_1}{2}}_{b_1-b_2 \text{ times}}, \underbrace{\frac{b_2-b_1}{2}, \dots, \frac{b_2-b_1}{2}}_{a_2-b_1+1 \text{ times}}, 0, \dots, 0)$$

First, $x_1 = \frac{b_1-b_2}{2}$. Secondly, since $x_k = \sum_{i=1}^k a_i$ as written in the proof of Lemma 32, one observes that all subsequent x_k are greater or equal to x_n, for $n = a_1 - b_1 + 1 + a_2 - b_2 + 1$.

And $x_n = \frac{b_1-b_2}{2}(a_1 - b_1 + 1) + \frac{a_2-a_1}{2}(b_1 - b_2) + \frac{b_2-b_1}{2}(a_2 - b_1 + 1) = \frac{b_1-b_2}{2}(a_1 - b_1 + 1 + a_2 - a_1 - (a_2 - b_1 + 1)) = 0$

— In case (2)

$$\lambda - \lambda' = (\underbrace{\frac{a_1 - a_2}{2}, \dots, \frac{a_1 - a_2}{2}}_{a_2-b_1+1 \text{ times}}, \underbrace{\frac{b_1 - b_2}{2}, \dots, \frac{b_1 - b_2}{2}}_{a_1-a_2 \text{ times}}, \underbrace{\frac{a_2 - a_1}{2}, \dots, \frac{a_2 - a_1}{2}}_{a_2-b_2+1 \text{ times}})$$

Here $x_1 = \frac{a_1-a_2}{2}$ $x_n = \frac{a_1-a_2}{2}(a_2 - b_1 + 1) + \frac{b_1-b_2}{2}(a_1 - a_2) + \frac{a_2-a_1}{2}(a_2 - b_2 + 1) = \frac{a_2-a_1}{2}(a_2 - b_1 + 1 + b_1 - b_2 - (a_2 - b_2 + 1)) = 0.$

□

Proposition 38. *The Langlands parameter λ', as defined in the previous Proposition 37, is the mimimal Langlands parameter for the order given in Lemma 31 on this cuspidal support.*

Proof. Let us consider a decreasing sequence of real numbers such that the difference between two consecutive elements is one : $(a_1, a_1 - 1, \dots, a_2, \dots, b_1, \dots, b_2)$ with the following conditions : $a_1 > a_2 > b_1 > b_2$ and all real numbers between a_2 and b_1 are repeated twice. Let us call this sequence c.

We consider the set $\mathscr{S}$ of tuple of linear segments $\mathcal{S}_i = (a_i, \dots, b_i)$ (stricly decreasing sequence of reals) such that if $s_i = \frac{a_i+b_i}{2} \geq s_j = \frac{a_j+b_j}{2}$ then the linear segment $\mathcal{S}_i$ is placed on the left of $\mathcal{S}_j$, i.e. :

$$(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k) \in \mathscr{S} \Leftrightarrow s_1 \geq s_2 \dots \geq s_k$$

In this set $\mathscr{S}$, let us first consider the special case of a decreasing sequence $\delta \in \mathscr{S}$ where each segment is length one and $s_i = \mathcal{S}_i$.

Then the Langlands parameter is just δ :

$$\delta = (a_1, a_1 - 1, \dots, a_2, a_2, \dots, b_1, b_1, \dots b_2)$$

Secondly, let us consider the case where all segments are mutually unlinked, then they have to be included in one another. The reader will readily notice that the only option is the following element in $\mathscr{S}$:

$$m := (a_1, \dots, b_2)(a_2, \dots, b_1)$$

Its Langlands parameter is :

$$\lambda' = (\underbrace{\frac{a_1 + b_2}{2}, \dots, \frac{a_1 + b_2}{2}}_{a_1-b_2+1 \text{ times}}, \underbrace{\frac{a_2 + b_1}{2}, \dots, \frac{a_2 + b_1}{2}}_{a_2-b_1+1 \text{ times}})$$

Let us show that $\delta \geq_P \lambda'$.

Clearly on the vector $\delta - \lambda' : x_1 = a_1 - \frac{a_1+b_2}{2} > 0$, $x_k = \sum_{i=1}^k a_i$ and one observes that all subsequent x_k are greater or equal to x_n, and x_n is the sum of the elements (counted with multiplicities) in the vector δ minus $\frac{a_1+b_2}{2}(a_1-b_2+1)+\frac{a_2+b_1}{2}(a_2-b_1+1)$, therefore $x_n = 0$ as this sum ends up the same as in the proof of the previous proposition.

Let us show that m is the unique, irreducible element obtained in $\mathscr{S}$ when taking repeatedly intersection and union of any two segments in any element $s \in \mathcal{S}$.

Let us write an arbitrary $s \in \mathscr{S}$ as $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_p)$, since we had a certain number of reals repeated twice in c, it is clear that some of the $\mathcal{S}_i$ are mutually linked.

For our purpose, we write the vector of lengths of the segments in $s : (k_1, k_2, \ldots, k_p)$.

Let us assume, without loss of generality, that $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked. Taking intersection and union, we obtain two unlinked segments $\mathcal{S}_1' = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_2' = \mathcal{S}_1 \cap \mathcal{S}_2$. If $k_1 \geq k_2$, then $k_1' = k_1 + a$, and $k_2' = k_2 - a$, i.e. the greatest length necessarily increases.

Therefore, the potential $\sum_i k_i^2$ is increasing, while the number of segments is non-increasing.

The process ends when we cannot take anymore intersection and union of linked segments, then the longest segment contains entirely the second longest, this is the element $m \in \mathscr{S}$ introduced above.

Since at each step (of taking intersection and union of two linked segments) the Langlands parameter $\lambda_{s'}$ of the element $s' \in \mathscr{S}$ is smaller than at the previous step (by Proposition 37), it is clear that λ' is the minimal element for the order on Langlands parameter.

□

Remark 6. Let us assume we fix the cuspidal representation σ and two segments $(\mathcal{S}_1, \mathcal{S}_2)$. As a result of this proposition, the standard module $I_{P'}^G(\tau'_{\lambda'})$ induced from the unique irreducible generic essentially square integrable representation $\tau'_{\lambda'}$ obtained when taking intersection and union $(\mathcal{S}_1 \cap \mathcal{S}_2)$ and $(\mathcal{S}_1 \cup \mathcal{S}_2)$(i.e. which embeds in $I_{P_1}^G(\sigma((\mathcal{S}_1 \cap \mathcal{S}_2); (\mathcal{S}_1 \cup \mathcal{S}_2)))$ is irreducible by Theorem 35.

1.6.3. Intertwining operators

In the following result, we play for the first time with cuspidal strings and intertwining operators. We fix a unitary irreducible cuspidal representation σ of M_1 and let $(a, b, \underline{n})$ and $(a', b', \underline{n'})$ be two elements in some W_σ-cuspidal string; i.e, there exists a Weyl group element $w \in W_\sigma$ such that $w(a, b, \underline{n}) = (a', b', \underline{n'})$.

For the sake of readability we sometimes denote $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is expressed in terms of residual segments. We would like to study intertwining operators between $I_{P_1}^G(\sigma(a, b, \underline{n}))$ and $I_{P_1}^G(\sigma(a', b', \underline{n'}))$. As explained in Section 1.3, Proposition 18, this operator can be decomposed in rank one operators. Let us recall how one can conclude on the non-genericity of their kernels in the two main cases.

Example 8 (Rank one intertwining operators with non-generic kernel)**.** Let us assume Σ_σ is irreducible of type A, B, C or D. We fix a unitary irreducible cuspidal representation σ and let $\alpha = e_i - e_{i+1}$ be a simple root in Σ_σ. The element s_α operates on λ in $(a_{M_1}^G)^*$. In this first example, we illustrate the case where s_α acts as a coordinates' transposition on λ written in the standard basis $\{e_i\}_i$ of $(a_{M_1}^G)^*$.

Let us focus on two adjacent elements in the residual segment corresponding to λ (at the coordinates λ_i and λ_{i+1}) : $\{a, b\}$, let us consider the rank one operator which goes from $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots\{a,b\}\ldots})$ to $I_{\overline{P_1}\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots\{a,b\}\ldots})$.

By Proposition 18 it is an operator with non-generic kernel if and only if $a < b$; Indeed if we denote $\lambda := (\ldots, a, b, \ldots)$, then $\langle\check{\alpha}, \lambda\rangle = a - b < 0$ (The action of s_α on λ leaves fixed the other coordinates of λ that we simply denote by dots).

Since $\alpha \in \Sigma_\sigma$, by point (a) in Harish-Chandra's Theorem [Theorem 8], there is a unique non-trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ such that $s_\alpha(P_1\cap(M_1)_\alpha) = \overline{P_1}\cap(M_1)_\alpha$ and which operates as the transposition from (a, b) to (b, a).

The rank one operator from $I_{\overline{P_1}\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,a,b,\ldots})$ to $I_{s_\alpha(\overline{P_1}\cap(M_1)_\alpha)}^{(M_1)_\alpha}(s_\alpha(\sigma_{\ldots,a,b,\ldots})) := I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,b,a,\ldots})$ is bijective.

Eventually we have shown that the composition of those two which goes from $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,a,b,\ldots})$ to $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,b,a,\ldots})$ has non-generic kernel.

If the Weyl group W_σ is isomorphic to $S_n \rtimes \{\pm 1\}$, the Weyl group element corresponding to $\{\pm 1\}$ is the sign change and we operate this sign change on the latest coordinate of λ (extreme right of the cuspidal string).

By the same argumentation as in the first example, for $a > 0$, the operator $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots-a})$ to $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots a})$ has non-generic kernel.

Example 9. Let G be a classical group of rank n. Let us take σ an irreducible unitary generic cuspidal representation of M_1, a standard Levi subgroup of G. Let us assume Σ_σ is irreducible of type B, and take $\lambda := (s_1, s_2, \ldots, s_m)$ in $a_{M_1}^*$, ρ an irreducible unitary cuspidal representation of GL_k, and σ_c an irreducible unitary cuspidal representation of $G(k')$ $k' < n$. Then σ_λ is :

$$\sigma_\lambda := \rho|.|^{s_1} \otimes \rho|.|^{s_2} \otimes \ldots \otimes \rho|.|^{s_m} \otimes \sigma_c$$

The element s_{α_i} operates as follows :

$$s_{\alpha_i}(\rho|.|^{s_1}\otimes\ldots\rho|.|^{s_i}\otimes\rho|.|^{s_{i+1}}\ldots\otimes\rho|.|^{s_m}\otimes\sigma_c) = \rho|.|^{s_1}\otimes\ldots\otimes\rho|.|^{s_{i+1}}\otimes\rho|.|^{s_i}\otimes\ldots\otimes\rho|.|^{s_m}\otimes\sigma_c$$

Indeed, for such α_i (which is in Σ_σ), one checks that property (a) in Theorem 8 holds : $s_{\alpha_i}(\sigma) \cong \sigma$. This is verified for any $i \in \{1, \ldots, n\}$. The intertwining operator usually considered in this manuscript is induced by functoriality from the application $\sigma_\lambda \to s_{\alpha_i}(\sigma_\lambda)$.

Lemma 39. *Let $b' \leq \ell+m, b \leq a$. Fix a unitary irreducible cuspidal representation σ of a maximal Levi subgroup in a quasi-split reductive group G, and two cuspidal strings $(a, b, \underline{n})$ and $(a, b', \underline{n'})$ in a W_σ-cuspidal string (notice that the right end of these are equals with value a). If $b' \geq b$, the intertwining operator between $I_{P_1}^G(\sigma(a, b, \underline{n}))$ and $I_{P_1}^G(\sigma(a, b', \underline{n'}))$ has non-generic kernel.*

Proof. In this proof, to detail the operations on cuspidal strings more explicitely we write the residual segments of type B, C, D defined in Definition 25 as :

$$((\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

where n_i denote the number of times the (half)-integer i is repeated. We present the arguments for integers, the proof for half-integers follows the same argumentation.

First, assume $b \geq 0$, and consider changes on the cuspidal strings

$$(a, \ldots, b', b'-1, \ldots b)((\ell+m)\ldots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\ldots b^{n_b}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

consisting in permuting successively all elements in $\{b, \ldots, b'-1\}$ with their right hand neighbor, as soon as this right hand neighbor is larger. We incorporate all elements starting with b until $b'-1$ from the left into the right hand residual segment. The rank one intertwining operators associated to those permutations have non-generic kernel (see Example 8) ; hence the intertwining operator from $I_{P_1}^G(\sigma(a, b, \underline{n}))$ to $I_{P_1}^G(\sigma(a, b', \underline{n'}))$ as composition of those rank one operators has non-generic kernel.

Assume now $b < 0$ and write $b = -\gamma$. Let us show that there exists an intertwining operator with non-generic kernel from the module induced from $I_{P_1}^G(\sigma(a, -\gamma, \underline{n}))$ to the one induced from $I_{P_1}^G(\sigma(a, b', \underline{n'}))$.

The decomposition in rank one operators has the following two steps (the details on the first step are given in the next paragraph) :

1. (a) If $b' \geq 1 > b$ From the cuspidal string

$$(a, \ldots, \gamma, \gamma-1, \ldots, -\gamma)((\ell+m)\ldots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\ldots b^{n_b}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

to

$$(a, \ldots, \gamma, \gamma-1, \ldots, 1)((\ell+m)\ldots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\ldots b^{n_b}\ldots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \ldots, -\gamma)$$

and then to

$$(a, \ldots, \gamma, \gamma-1, \ldots, 1)((\ell+m)\ldots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\ldots b^{n_b+1}\ldots 2^{n_2+1}1^{n_1+1}0^{n_0+1})$$

(b) If $0 \geq b' \geq b$ From the cuspidal string

$$(a, \ldots, \gamma, \gamma - 1, \ldots, -\gamma)((\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b} \ldots 2^{n_2} 1^{n_1} 0^{n_0})$$

to

$$(a, \ldots, \gamma, \gamma - 1, \ldots, b')((\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b} \ldots 2^{n_2} 1^{n_1} 0^{n_0+1}, b' - 1, \ldots, -\gamma)$$

and then to

$$(a, \ldots, \gamma, \gamma - 1, \ldots, b')((\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b+1} \ldots 2^{n_2+1} 1^{n_1+1} 0^{n_0+1})$$

2. In case (a), from $(a, \ldots, 1)(\underline{n''})$ to $(a, \ldots, b')(\underline{n'})$ by the same arguments as in the case $b \geq 0$ treated in the first paragraph of this proof.

We detail the operations in step 1 :

(i) Starting with $-\gamma$, all negative elements in $\{0, \ldots, -\gamma\}$ are successively sent to the extreme right of the second residual segment $(\underline{n})$. At each step, the rank one intertwining operator between (a, p) and (p, a) where p is a negative integer (or half-integer) and $a > p$ has non-generic kernel.

(ii) We use rank one operators of the second type (sign chance of the extreme right element of the cuspidal string). Since they intertwine cuspidal strings where the last element changes from negative to positive, they have non-generic kernels. Then, the positive element is moved up left. The right-hand residual segment goes from

$$((\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b} \ldots 2^{n_2} 1^{n_1} 0^{n_0+1}, -1, \ldots, -\gamma)$$

to

$$((\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b} \ldots 2^{n_2} 1^{n_1} 0^{n_0+1}, -1, \ldots, \gamma)$$

and then to

$$((\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b} \ldots 2^{n_2} 1^{n_1} 0^{n_0+1}, \gamma, -1, \ldots, -(\gamma - 1))$$

Once changed to positive, permuting successively elements from right to left, one can reorganize the residual segment $(\ell + m) \ldots \ell^{n_\ell} (\ell - 1)^{n_{\ell-1}} \ldots b^{n_b} \ldots 2^{n_2} 1^{n_1} 0^{n_0+1}, \gamma, \ldots 1)$ such as it is a decreasing sequence of (half)-integers. Again intertwining operators following these changes on the cuspidal string have non-generic kernels.

□

Example 10. Consider the cuspidal string (543210-1)(43 322 211 1 0) and the dominant residual point in its W_σ-cuspidal string : (54 433 3222 21111 10 0). To the Weyl group element $w \in W_\sigma$ associate an intertwining operator from the module induced with string (534210-1)(43 322 211 1 0) to the one induced with cuspidal-string (54 433 3222 21111 10 0) which has non-generic kernel.

Indeed one will decompose it into transpositions s_α such as (-1,4) to (4,-1) and similarly for any $4 > i \geq 0 : (-1, i)$ to $(i, -1)$.

This process will result in (543210)(43 322 211 1 0 -1). Then one will change the -1 to 1, and by the above the associated rank-one operator also has non-generic kernel.

Then notice that the '4', '3' and '2' in the middle of the sequence can be moved to the left with a sequence of rank one operators with non- generic kernel such as :$(0, 4) \to (4, 0); \ldots; (3, 4) \to (4, 3)$.

Lemma 40. *Let $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t)$ be an ordered sequence of t linear segments and let us denote $\mathcal{S}_i = (a_i, \ldots, b_i)$, for any i in $\{1, \ldots, t\}$. This sequence is ordered so that for any i in $\{1, \ldots, t\}$, $s_i = \frac{a_i + b_i}{2} \geq s_{i+1} = \frac{a_{i+1} + b_{i+1}}{2}$. Let us assume that for some indices in $\{1, \ldots, t\}$ the linear residual segments are linked.*

Let us denote $(\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t)$ the ordered sequence corresponding to the end of the procedure of taking union and intersection of linked linear residual segments. This sequence is composed of at most t unlinked residual segments $\mathcal{S}'_i = (a'_i, \ldots, b'_i), i \in \{1, \ldots, t\}$.

Taking repeatedly intersection and union yields smaller Langlands parameters for the order defined in Lemma 31 ; and we denote the smallest element for this order, $\underline{s'}$. It corresponds to the sequence $(\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t)$ as explained in Lemma 38.

Then there exists an intertwining operator with non-generic kernel from the induced module $I_{P_1}^G(\sigma((\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t; \underline{n}))$ to $I_{P_1}^G(\sigma((\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t; \underline{n}))$.

Proof. Let us first consider the case $t = 2$.

Consider two linear (i.e of type A) residual segments, i.e stricly decreasing sequences of either consecutive integers or consecutive half-integers $\mathcal{S}_1 := (a_1, \ldots, b_1); \mathcal{S}_2 := (a_2, \ldots, b_2)$.

Assume $a_1 > a_2 > b_1 > b_2$ so that they are linked in the terminology of Bernstein-Zelevinsky. Taking intersection and union yield two unlinked linear residual segments $\mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathcal{S}_1 \cup \mathcal{S}_2 : (a_1, \ldots, b_2)(a_2, \ldots, b_1)$ or $(a_2, \ldots, b_1)(a_1, \ldots, b_2)$ ordered so that $s'_1 > s'_2$.

As in the proof of Lemma 39, because $a_2 > b_2$ and also $b_1 > b_2$ there exists an interwining operator with non-generic kernel from the module induced with cuspidal support $(a_1, \ldots, b_2)(a_2, \ldots, b_1)$ to the one induced with cuspidal support $(a_1, \ldots, b_1)(a_2, \ldots, b_2)$.

This intertwining operator is a composition of rank one intertwining operators associated to permutations which have non-generic kernel (see Example 8) ; as composition of those rank one operators, it has non-generic kernel.

Similarly, because $a_1 > a_2$, there exists an interwining operator with non-generic kernel from the module induced with cuspidal support $(a_2, \ldots, b_1)(a_1, \ldots, b_2)$ to the one induced with cuspidal support $(a_1, \ldots, b_1)(a_2, \ldots, b_2)$.

Let us now assume the result of this lemma true for t linear residual segments.

Consequently, there exists an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t, \mathcal{S}_{t+1}), \underline{n})$ to $I_{P_1}^G(\sigma((\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t, \mathcal{S}_{t+1}), \underline{n})$. In this case

$\mathcal{S}_{t+1}$ and $\mathcal{S}'_t$ may be linked and taking union and intersection of them yields $\mathcal{S}'_{t+1}$ and $\mathcal{S}''_t$ and the existence of an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}''_t, \mathcal{S}'_{t+1}), \underline{n})$ to $I_{P_1}^G(\sigma((\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t, \mathcal{S}_{t+1}), \underline{n})$. The latter argument is repeated if $\mathcal{S}''_t$ and $\mathcal{S}'_{t-1}$ are linked, and so on. Eventually there exists an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}^*_1, \mathcal{S}^*_2, \ldots, \mathcal{S}^*_t, \mathcal{S}^*_{t+1}; \underline{n}))$ to $I_{P_1}^G(\sigma((\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t, \mathcal{S}_{t+1}; \underline{n}))$, where $(\mathcal{S}^*_1, \mathcal{S}^*_2, \ldots, \mathcal{S}^*_t, \mathcal{S}^*_{t+1})$ is the sequence of $t+1$ unlinked segments obtained at the end of the procedure of taking intersection and union. $\square$

1.7. Generalized Injectivity conjecture for Σ_σ of type A

1.7.1. A Lemma in the vein of Zelevinsky's Theorem

Recall this fundamental result of Zelevinsky, for the general linear group, which was also presented as Theorem 5 in Rodier 1981-1982. We use the notation introduced in Definition 26.

Proposition 41 (Zelevinsky, Zelevinsky 1980, Theorem 9.7)**.** *If any two segments, $\mathcal{S}_i, \mathcal{S}_j$, j, i in $\{1, \ldots, n\}$ of the linear group are not linked, we have the irreducibility of $Z(\mathcal{S}_1) \times Z(\mathcal{S}_2) \times \ldots \times Z(\mathcal{S}_n)$ and conversely if $Z(\mathcal{S}_1) \times Z(\mathcal{S}_2) \times \ldots \times Z(\mathcal{S}_n)$ is irreducible, then all segments are mutually unlinked.*

Here, we prove a similar statement in the context of any quasi-split reductive group of type A.

Lemma 42. *Let τ be an irreducible generic discrete series of a standard Levi subgroup M in a quasi-split reductive group G. Let σ be an irreducible unitary generic cuspidal representation of a standard Levi subgroup M_1 in the cuspidal support of τ. Let us assume Σ_σ is irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$ and type A.*

Let $\underline{s} = (s_1, s_2, \ldots, s_t) \in a^*_{M_1}$ be ordered such that $s_1 \geq s_2 \geq \ldots \geq s_t$ with $s_i = \frac{a_i + b_i}{2}$, for two real numbers $a_i \geq b_i$.

Then $I_P^G(\tau_{\underline{s}})$ is a generic standard module embedded in $I_{P_1}^G(\sigma_\lambda)$ and λ is composed of t residual segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ of type A_{n_i}.

Let us assume that the t segments are mutually unlinked. Then λ is not a residual point and therefore the unique irreducible generic subquotient of the generic module $I_{P_1}^G(\sigma_\lambda)$, is not a discrete series. This irreducible generic subquotient is $I_P^G(\tau_{\underline{s}})$. In other words, the generic standard module $I_P^G(\tau_{\underline{s}})$ is irreducible. Further, for any reordering $\underline{s'}$ of the tuple $\underline{s}$, which corresponds to an element $w \in W$ such that $w\underline{s} = \underline{s'}$ and discrete series τ' of M' such that $w\tau = \tau'$, $wM = M'$. $I_{P'}^G(\tau'_{\underline{s'}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$.

Proof. By the result of Heiermann-Opdam (Proposition 12), there exists a standard parabolic subgroup P_1, a unitary cuspidal representation σ, a parameter $\nu \in \overline{(a_{M_1}^{M} *)^{+}}$ such that the generic discrete series τ embeds in $I_{M_1 \cap M}^{M}(\sigma_\nu)$. By Heiermann's Theorem (see Theorem 11), ν is a residual point so it is composed of residual segments of type A_{n_i}. Then twisting by $\underline{s}$ and inducing to G, we obtain :

$$I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda) \text{ where } \lambda = (a_i, \ldots, b_i)_{i=1}^t$$

Let π be the unique irreducible generic subquotient of the generic standard module $I_P^G(\tau_{\underline{s}})$. Then using Langlands' classification and the standard module conjecture $\pi = J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'})$.

Assume τ' is discrete series. We apply again the result of Heiermann-Opdam to this generic discrete series to embed $I_{P'}^G(\tau'_{\nu'})$ in $I_{P'_1}^G(\sigma'_{\lambda'})$.

As any representation in the cuspidal support of $\tau_{\underline{s}}$ must lie in the cuspidal support of π, any such representation much be conjugated to $\sigma'_{\lambda'}$, therefore λ' is in the Weyl group orbit of λ. Let us consider this Weyl group orbit under the assumption that the t segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ are unlinked.

Whether the union of any two segments in $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ is not a segment, or the segments are mutually included in one another, it is clear there are no option to take intersections and unions to obtain new linear residual segments. Further, starting with λ, to generate new elements in its Weyl group orbit, one can split the segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$. By Lemma 36, this procedure yields necessarily larger Langlands parameters. Therefore there is no option to reorganize them to obtain residual segments (a'_j, b'_j) of type $A_{n'_j}$ such that $n'_j \neq n_i$ for some $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, for some r such that $\sum_{j=1}^{r} n'_j = \sum_{i=1}^{t} n_i$.

The second option is to permute the order of the segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ to obtain any other parameter λ' in the Weyl group orbit of λ. From this λ', one clearly obtains the parameter $\nu' := \underline{s'}$ as a simple permutation of the tuple $\underline{s}$.

On the Langlands parameter $\underline{s}$, which is the unique among the (ν')'s described in the previous paragraph in the Langlands situation (we consider all standard modules $I_{P'}^G(\tau'_{\nu'})$), we can use Theorem 35 to conclude that the generic standard module $I_P^G(\tau_{\underline{s}})$ for $\nu = \underline{s}$ is irreducible.

Now, we want to show $I_{P'}^G(\tau'_{\underline{s'}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$.

Looking at the cuspidal support, it is clear that there exists a Weyl group element in $W(M, M')$ sending σ_λ to $\sigma'_{\lambda'}$, and therefore $\tau_{\underline{s}}$ to the Langlands data $(w\tau)_{w\underline{s}} := \tau'_{\underline{s'}}$.

Consider first the case of a maximal parabolic subgroup P in G. Set $\underline{s} = (s_1, s_2)$, $\underline{s'} = (s_2, s_1)$ and τ' is a generic discrete series representation. We apply the map $t(w)$ between $I_P^G(\tau_{\underline{s}})$ and $I_{wP}^G((w\tau)_{w\underline{s}})$ which is an isomorphism. By definition, the parabolic wP has Levi M'. Then, by the Theorem 2.9 in Bernstein et Zelevinsky 1977 (see also Renard 2010 VI.5.4), since the Levi subgroups and inducing representations are the same, the Jordan-Hölder composition series of $I_{wP}^G(\tau'_{\underline{s'}})$ and $I_{P'}^G(\tau'_{\underline{s'}})$ are the same, and since $I_P^G(\tau_{\underline{s}})$ is irreducible, they are isomorphic and irreducible.

Secondly, consider the case when the two parabolic subgroups P and P', with Levi subgroup M and M', are connected by a sequence of adjacent parabolic subgroups of G.

Using Theorem 35 with any Levi subgroup in G, in particular a Levi subgroup M_α (containing M as a maximal Levi subgroup) shows that the representation $I^{M_\alpha}_{P\cap M_\alpha}(\tau_{\underline{s}})$ is irreducible.

Then, we are in the context of the above paragraph and $I^{M_\alpha}_{s_\alpha(\overline{P}\cap M_\alpha)}((s_\alpha\tau)_{s_\alpha\underline{s}})$ (the image of the composite of the map $J_{\overline{P}\cap M_\alpha|P\cap M_\alpha}$ with the map $t(s_\alpha)$) is irreducible, and isomorphic to $I^{M_\alpha}_{P\cap M_\alpha}(\tau_{\underline{s}})$.

Let us denote Q the parabolic subgroup adjacent to P along α. Induction from M_α to G yields that $I^G_Q(s_\alpha\tau)_{s_\alpha\underline{s}})$ is isomorphic to $I^G_P(\tau_{\underline{s}})$. Writing the Weyl group element w in $W(M,M')$ such that $wM = M'$ as a product of elementary symmetries s_{α_i}, and applying a sequence of intertwining maps as above yields the isomorphism between $I^G_P(\tau_{\underline{s}})$ and $I^G_{P'}(\tau'_{\underline{s'}})$. $\square$

Example 11 (See Bernstein et Zelevinsky 1977, 2.6)**.** Let $W^G = N_G(A_0)/Z_G(A_0)$ for a maximal split torus A_0 in G. Let M and N be standard Levi subgroups of G. We set $W(M,N) = \left\{w \in W^G | w(M) = N\right\}$; it is clear that $W^N.W(M,N).W^M = W(M,N)$.

The subgroups M and N are associated (the notation $M \sim N$) if $W(M,N) \neq \emptyset$.

Any element $w \in W(M,N)$ determines the functor $w : AlgM \to AlgN$; and representation $\rho \in AlgM, \rho' \in AlgN$ are called associated if $\rho' \cong w\rho$ for $w \in W(M,N)$ (the notation $\rho \sim \rho'$).

Let $G = G_n = GL_n, \alpha = (n_1, \dots, n_r)$ and $\beta = (n'_1, \dots, n'_s)$ be partitions of n. To each partition $\alpha = (n_1, \dots, n_r)$ corresponds the standard Levi subgroup $G_\alpha = G_{n_1} \times G_{n_2} \times \dots \times G_{n_r}$. Set $M = G_\alpha, N = G_\beta$. Then the condition $M \sim N$ means $r = s$ and the family $(n_1, \dots, n_r)$ is a permutation of $(n'_1, \dots, n'_s)$. Such permutation corresponds to elements of $W(M,N)/W^N$. Let $\rho_i \in IrrG_{n_i}, \rho'_i \in IrrG_{n'_i}, \rho = \otimes\rho_i \in IrrM$, and $\rho' = \otimes\rho'_i \in IrrN$, then $\rho \sim \rho'$ iff the set $(\rho_1, \dots, \rho_r)$ and $(\rho'_1, \dots, \rho'_r)$ are equal up to permutation.

Theorem 43. *Let G be a quasi-split reductive group of type A, B, C or D. Let P be a standard parabolic subgroup $P = MU$ of G.*

Let us consider $I^G_P(\tau_{\underline{s}})$ with τ an irreducible discrete series of M, $\underline{s} \in (a^*_M)^+$. Let σ be a unitary cuspidal representation of M_1 in the cuspidal support of τ and assume Σ_σ (defined with respect to G) is of type A and irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$.

A typical example is when G is of type A.

Then the unique irreducible generic subquotient of $I^G_P(\tau_{\underline{s}})$ is a subrepresentation.

Proof. As proven in Appendix E, when G is of type A, if Σ_σ is irreducible of rank d, it is necessarily of type A.

Let τ be a discrete series of M a standard Levi subgroup of G. By the result of Heiermann-Opdam [Proposition 12], there exists a standard parabolic subgroup P_1 such that $\tau \hookrightarrow I_{P_1}^M(\sigma_\nu)$ with ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$. We consider the unique irreducible generic subquotient in the standard module $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$. We denote $\lambda = \nu + \underline{s}$.

Let us first consider the case of M being a maximal Levi subgroup in G. M is obtained by removing a (non-extremal) root from Δ, and therefore obtain two subsystems of roots of type A_{i-1} and A_{d-i} in Σ_σ^M.

The character ν is constituted of two residual segments of type A_{i-1} and A_{d-i} : (ν_1, ν_2) and when we twist by $s\tilde{\alpha}$ we obtain the Langlands parameter $\lambda = (\nu_1 + s_1, \nu_2 + s_2)$ where $s_1 = \frac{a_1+b_1}{2} > s_2 = \frac{a_2+b_2}{2}$ and write $\lambda = (a_1, b_1; a_2, b_2)$ for two residual segments (a_j, b_j) of type A.

Assume σ_λ is a residual point, that is the sequence $(a_1, b_1; a_2, b_2)$ shall be a strictly decreasing sequence of real numbers (corresponding to a segment of type A_d).

This means $b_1 = a_2 + 1$, and therefore σ_λ is already in a dominant position with respect to P_1. So λ is a dominant residual point and therefore by Lemma 34 the unique irreducible generic discrete series subquotient embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and consequently (by unicity of the generic irreducible piece) in the standard module $I_P^G(\tau_{\underline{s}})$.

Else, σ_λ is not a residual point.

In this case, either the two segments $(a_1, \ldots, b_1); (a_2, \ldots, b_2)$ are unlinked and therefore, by Lemma 42, the standard module $I_P^G(\tau_{\underline{s}})$ is irreducible ; or the two segments are linked and the sequence $(a_1, b_1; a_2, b_2)$ can be reorganized in only one way to obtain two residual segments of type $A_{i'}$ and $A_{n-i'}$ which is to take intersection and union of $(a_1, \ldots, b_1)$ and $(a_2, \ldots, b_2)$. We obtain two unlinked segments $(a_2, \ldots, b_1) \subset (a_1, \ldots, b_2)$ [see the Definition 26] and by Lemma 37 the Langlands parameter (this notion was introduced in the Theorem 13) $\underline{s'} := (s_1', s_2')$ is smaller than (s_1, s_2).

By Proposition 37, the parameter $\underline{s'}$ is the minimal element for the order defined in Lemma 31. Let M' be the maximal Levi subgroup which corresponds to removing the root from the A_d Dynkin diagram to obtain $A_{i'-1}$ and $A_{d-i'}$. Let τ' be the discrete series subrepresentation of $I_{P_1 \cap M'}^{M'}(\sigma((\frac{a_2-b_1}{2}, \ldots, -\frac{a_2-b_1}{2})(\frac{a_1-b_2}{2}, \ldots, -\frac{a_1-b_2}{2})))$.

Since $\underline{s'}$ is the minimal element for the order defined in Lemma 31, by Theorem 35, the module $I_{P'}^G(\tau'_{\underline{s'}})$ is the unique irreducible generic subquotient of $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Further, notice that in Lemma 42, $\underline{s'}$ can be ordered as one wishes, or said differently we order the residual segments $(a_2, \ldots, b_1) \subset (a_1, \ldots, b_2)$ as one wishes. Then we can embed as a subrepresentation $I_{P'}^G(\tau'_{\underline{s'}})$ in $I_{P_1}^G(\sigma(a_2, b_1, a_1, b_2))$.

We now consider the intertwining operator from $I_{P_1}^G(\sigma(a_2, b_1, a_1, b_2))$ to $I_{P_1}^G(\sigma((a_1, b_1, a_2, b_2))$. Since $a_1 \geq a_2$, we can use Lemma 39 to conclude that it has non-generic kernel. Therefore $I_{P'}^G(\tau'_{\underline{s'}})$ embeds as a subrepresentations in $I_{P_1}^G(\sigma(a_1, b_1, a_2, b_2))$ and therefore in $I_P^G(\tau_{\underline{s}})$ by unicity of the generic piece in the induced representation $I_{P_1}^G(\sigma((a_1, b_1, a_2, b_2))$.

Secondly, consider the case of a non-necessarily maximal standard Levi subgroup, then we have t subsystems of type A.

If $\nu + \underline{s} = \lambda := (\bigotimes(a_i, \ldots, b_i)_{i=1}^t)$ is a residual point, it shall be a decreasing sequence of real numbers, therefore in dominant position, and we can immediately conclude by Lemma 34 and the unique irreducible generic discrete series subquotient embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and therefore in $I_P^G(\tau_{\underline{s}})$ by unicity of the generic piece.

Else, λ is not a residual point, and therefore the unique irreducible generic subquotient reads $I_{P'}^G(\tau'_{\underline{s'}})$ where $\underline{s'}$ is the smallest Langlands parameter with respect to the order defined in Lemma 31 on Langlands parameters. If all the linear residual segments $\{(a_i, \ldots, b_i)\}_{i=1}^t$ are unlinked, by Lemma 42, the standard module $I_P^G(\tau_{\underline{s}})$ is irreducible.

Otherwise, let us assume that for some indices i, j in $\{1, \ldots, t\}$, the two linear segments $(a_i, \ldots, b_i)$ and $(a_j, \ldots, b_j)$ are linked.

By Proposition 37, $\underline{s'} = (s'_1, s'_2, \ldots, s'_t) < \underline{s}$ if it is obtained by taking repeatly intersection and union of all two linked linear segments (at each step taking intersection and union of two segments and leaving the other segments unchanged gives a smaller Langlands parameter by Proposition 37).

Let us denote $w\lambda = (\bigotimes(a'_i, \ldots, b'_i)_{i=1}^t)$ (for some Weyl group element w in W_σ) the parameter obtained by taking repeatly intersection and union of all two linked linear segments.

Further from Lemma 42, $\underline{s'}$ can be ordered as one wishes, or said differently how we order the t unlinked residual segments (a'_i, b'_i)'s in the parameter $w\lambda = ((a'_i, \ldots, b'_i))_{i=1}^t$ does not matter. Now, the irreducible generic discrete series τ' is (by the result of Heiermann-Opdam, Proposition 12) a subrepresentation in $I_{P_1 \cap M'}^{M'}(\sigma(\oplus_{i=1}^t(\frac{a'_i - b'_i}{2}, -\frac{a'_i - b'_i}{2})))$.

Then, $I_{P'}^G(\tau'_{\underline{s'}})$ embeds as a subrepresentation in $I_{P_1}^G((w\sigma)_{w\lambda}) = I_{P_1}^G(\sigma_{w\lambda})$. The last equality because $w\sigma \cong \sigma$.

Further, there will exist a certain order on the unlinked residual segments $(a'_i, \ldots, b'_i)$ allowing the existence of an intertwining operator with non- generic kernel from $I_{P_1}^G(\sigma_{w\lambda})$ to $I_{P_1}^G(\sigma_\lambda)$ using repeatedly Lemma 39. Therefore the generic module $I_{P'}^G(\tau'_{\underline{s'}})$ appears as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and therefore in $I_P^G(\tau_{\underline{s}})$. □

1.8. Proof of the Generalized Injectivity Conjecture for Discrete Series Subquotients

In order to use Theorem 11, let us first prove the following lemma :

Lemma 44. *Under the assumption that μ^G has a pole in $s\tilde{\alpha}$ (assumption 1) for τ and μ^M has a pole in ν (for σ) of maximal order, for $\nu \in a_{M_1}^*$, $\sigma_{\nu+s\tilde{\alpha}}$ is a residual point.*

Proof. We will use the multiplicativity formula for the μ function (see Section IV 3 in Waldspurger 2003, or the earlier result (Theorem 1) in Silberger 1980a) :

$$\mu^G(\tau_{s\tilde{\alpha}}) = \frac{\mu^G}{\mu^M}(\sigma_{s\tilde{\alpha}+\nu})$$

We first notice that if μ^M has a pole in ν (for σ) of maximal order, for $\nu \in a^*_{M_1}$, μ^M also has a pole of maximal order in $\nu + s\tilde{\alpha}$ (Since $s\tilde{\alpha}$ is in a^*_M, we twist by a character of A_M which leaves the function μ^M unchanged). Under the assumption 1, the order of the pole in $\nu + s\tilde{\alpha}$ of the right side of the equation is :

$$\text{ord}(\text{pole for } \mu^G \text{ in } \nu + s\tilde{\alpha}) - (rk_{ss}(M) - rk_{ss}(M_1)) \geq 1$$

Since M is maximal we have : $(rk_{ss}(G) - rk_{ss}(M)) = \dim(A_M) - \dim(A_G) = 1$, then

$$\begin{aligned}(rk_{ss}(M) - rk_{ss}(M_1)) + 1 &= (rk_{ss}(M) - rk_{ss}(M_1)) + (rk_{ss}(G) - rk_{ss}(M)) \\ &= (rk_{ss}(G) - rk_{ss}(M_1))\end{aligned}$$

Hence $\text{ord}(\text{pole of } \mu^G \text{ in } \nu + s\tilde{\alpha}) \geq (rk_{ss}(G) - rk_{ss}(M_1))$, and the lemma follows. □

The element $\nu + s\tilde{\alpha}$ being a residual point (a pole of maximal order for μ^G) for σ, by Theorem 11 we have a discrete series subquotient in $I^G_{P_1}(\sigma_{\nu+s\tilde{\alpha}})$.

Further, consider the following classical lemma (see for instance Zhang 1997) :

Lemma 45. *Take τ a tempered representation of M, and ν_0 in the positive Weyl chamber. If ν_0 is a pole for μ^G then $I^G_P(\tau_{\nu_0})$ is reducible.*

This lemma results from the fact that when τ is tempered and ν_0 in the positive Weyl chamber, $J_{\overline{P}|P}(\tau, .)$ is holomorphic at ν_0. If the μ function has a pole at ν_0 then $J_{\overline{P}|P}J_{P|\overline{P}}(\tau, .)$ is the zero operator at ν_0. The image of $J_{P|\overline{P}}(\tau, .)$ would then be in the kernel of $J_{\overline{P}|P}(\tau, .)$, a subspace of $I^G_P(\tau_{\nu_0})$ which is null if $I^G_P(\tau_{\nu_0})$ is irreducible. This would imply $J_{P|\overline{P}}$ is a zero operator which is not possible. So $I^G_P(\tau_{\nu_0})$ must be reducible.

Under the hypothesis of Lemma 44, the module $I^G_P(\tau_{s\tilde{\alpha}})$ has a generic discrete series subquotient. We aim to prove in this section that this generic subquotient is a subrepresentation.

1.8.1. An embedding property for the generic discrete series subquotient

We use the notations and context introduced in Section 1.3.

Let γ be a generic discrete series subquotient, λ' be the dominant residual point in the Weyl group orbit of $\lambda = \nu + s\tilde{\alpha}$, we have the following diagram :

$$
\begin{tikzcd}
\gamma \leq & I_P^G(\tau_{s\tilde{\alpha}}) \arrow[r, hook] & I_{P_1}^G(\sigma_\lambda) \arrow[d, dotted, "\text{non-generic kernel}"] \\
\gamma \arrow[rr, hook] & & I_{P'}^G(\sigma'_{\lambda'}) \arrow[u, dotted, bend left, "\text{generic kernel}"]
\end{tikzcd}
$$

The Heiermann-Opdam Proposition 12, giving the embedding in $I_{P'}^G(\sigma'_{\lambda'})$ with λ' dominant residual point, although insufficient to immediately use intertwining operators and conclude, will also give us another embedding in a module $I_{P''}^G(\sigma''_{\lambda''})$ where λ'' is characterized using the language of residual segments and the notion of *Jumps* of residual segments as explained in the Subsection 1.5.2. This embedding result was formulated as Proposition 3.1 in Hanzer 2010, although the proof of the proposition there relies on both nomenclature and results of Moeglin et Tadic 2002.

From here, we will use the following notations :

Notations. — *For the sake of readability we sometimes denote* $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ *when the parameter* λ *is expressed in terms of residual segments.*

— *Let* σ *be an irreducible cuspidal representation of a Levi subgroup* $M_1 \subset M$ *in a standard parabolic subgroup* P_1*, and let* λ *be in* $(a_{M_1}^*)$*, we will denote* $Z^M(P_1, \sigma, \lambda)$ *the unique irreducible generic discrete series (resp. essentially square-integrable) in the standard module* $I_{P_1 \cap M}^M(\sigma_\lambda)$*.*

We will omit the index when the representation is a representation of G *:* $Z(P_1, \sigma, \lambda)$ *; often* λ *will be written explicitely with residual segments to emphasize the dependency on specific sequences of exponents.*

Let G be a quasi-split reductive group over F (resp. a product of group) whose root system Σ is of type A, B, C or D, π_0 is an irreducible generic discrete series of G whose cuspidal support contains the representation σ_λ of a standard Levi subgroup M_1, where $\lambda \in a_{M_1}^*$ and σ is an irreducible unitary cuspidal generic representation.

Let

$$d = rk_{ss}(G) - rk_{ss}(M_1) = \dim a_{M_1} - \dim a_G$$

Let us denote $M_1 = M_\Theta$. Then $\Delta - \Theta$ contains d simple roots.

Let us denote $\Delta(P_1)$ the set of non-trivial restrictions (or projections) to A_{M_1} (resp. to $a_{M_1}^G$) of simple roots in Δ such that elements in $\Sigma(P_1)$ (roots which are positive for P_1) are linear combinations of simple roots in $\Delta(P_1)$.

Let us denote $\Delta(P_1) = \{\alpha_1, \ldots, \alpha_{d-1}, \beta_d\}$ and $\underline{\alpha}_i$ the simple root in Δ which projects onto α_i in $\Delta(P_1)$.

As (M_1, σ_λ) is the cuspidal support of an irreducible discrete series, as explained in the Proposition 24), the set Σ_σ is a root system of rank d in $\Sigma(A_{M_1})$ and its basis, when we set $\Sigma(P_1) \cap \Sigma_\sigma$ as the set of positive roots for Σ_σ, is Δ_σ.

Proposition 46. *With the context of the previous paragraphs. Let Σ_σ be irreducible. If $\Delta(P_1) = \{\alpha_1, \ldots, \alpha_{d-1}, \beta_d\}$ then $\Delta_\sigma = \{\alpha_1, \ldots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D.*

Proof. This is a result of the case-by-case analysis conducted in Appendix E, where Δ_Θ denotes there the $\Delta(P_1)$ considered in this Proposition. From its definition Σ_σ is a subsystem in Σ_Θ. If Σ_Θ contains a root system of type BC_d, it is clear that the last root, denoted α_d, of this system (which is either the short of long root depending on the chosen reduced system) can be different from β_d if Σ_σ is of type D_d. □

We have not included the root β_d in Δ_σ because (as opposed to the context of classical groups) it is possible that there exists σ an irreducible cuspidal representation such that $s_{\beta_d}\sigma \ncong \sigma$.

A typical example of the above Proposition (46) is when Σ if of type B, C and Σ_σ is of type D, then it occurs that $\Delta(P_1)$ contains $\beta_d = e_d$ or $\beta_d = 2e_d$ whereas Δ_σ contains $\alpha_d = e_{d-1} + e_d$.

This proposition allows us to use our results on intertwining operators with non-generic kernel (see Proposition 18, and Example 8).

In the context of Harish-Chandra's Theorem 8, the element denoted s_α corresponds to the element $\widetilde{w_0}^{(M_1)_\alpha}\widetilde{w_0^{M_1}}$ as defined in Chapter 1 in Shahidi 2010.

Let us describe it :

Let P be a standard parabolic, $P = MN$. Let $\Theta \subset \Delta$, $M = M_\Theta$. In Shahidi 2010, Shahidi defines $\widetilde{w_0}$ as the element in $W(A_0, G)$ which sends Θ to a subset of Δ but every other root $\beta \in \Delta - \Theta$ to a negative root.

If $\widetilde{w_0^G}$, $\widetilde{w_0^M}$ are the longest elements in the Weyl groups of A_0 in G and M, respectively, then $\widetilde{w_0} = \widetilde{w_0^G}\widetilde{w_0^M}$.

The length of this element in W is the difference of the lengths of each element in this composition. Therefore, if a representative of this element in G normalizes M, since it is of minimal length in its classe in the quotient $\{w \in W | w^{-1}Mw = M\} / W^M$, this representative belongs to $W(M)$.

When P is maximal and self-associate (meaning $\widetilde{w_0}(\Theta) = \Theta$) then if α is the simple root of A_M in Lie(N), $\widetilde{w_0}(\alpha) = -\alpha$.

In this case $w_0 N w_0^{-1} = N^-$, the opposite of N for w_0 a representative of $\widetilde{w_0}$ in G.

Remark 7. Applying the previous paragraph to the context of $P_1 \cap (M_1)_\beta$ and $(M_1)_\beta$, we first show that $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}}(\Theta) = \Theta$. Then, one notices that $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}}$ sends β to $-\beta$.

In analogy with the notations of Theorem 8, let us denote $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}} = s_\beta$, we have : $s_\beta(P_1 \cap (M_1)_\beta) = \overline{P_1} \cap (M_1)_\beta$, then $s_\beta\lambda = \lambda$ if λ is in $\overline{(a_{M_1}^G *)^+}$ and is a residual point of type D.

By definition, if $\alpha \in \Sigma_\sigma$, by Harish-Chandra's Theorem 8, $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$ and $s_\alpha.M_1 = M_1$, and this means that s_α is a representative in G of a Weyl group element sending Θ on Θ.

Corollary 47. *Let σ be an irreducible cuspidal representation of a standard Levi subgroup M_1 and let us assume that Σ_σ is irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$ and type A, B, C or D, then :*

1. *For any α in $\Delta(P_1)$, $s_\alpha \in W(M_1)$.*
2. *$W(M_1) = W_\sigma \cup \{s_{\beta_d} W_\sigma\}$.*
3. *Let σ' (resp. σ) be an irreducible cuspidal representation of a standard Levi subgroup M_1' (resp. standard Levi subgroup M_1). Let us assume they are the cuspidal support of the same irreducible discrete series. Then $M_1' = M_1$.*

Proof. **Point (1) :**
Let us assume Θ has the form given in Appendix E, Theorem 88, that is a disjoint union of irreducible components : $\bigcup_{i=1}^n \Theta_i$. Then, let us show that for any α in $\Delta(P_1)$, $s_\alpha \in W(M_1)$.

By definition, s_α is a representative in G of the element $\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}$.
Let us first assume that α_i is the restriction of the simple root connecting Θ_i and Θ_{i+1}, both of type A, in the Dynkin diagram of G.
Then

$$\Delta^{(M_1)_{\alpha_i}} = \Theta_i \cup \{\alpha_i\} \cup \Theta_{i+1} \bigcup_{j \neq i,i+1} \Theta_j$$

The element $\widetilde{w_0^{M_1}}$ operates on each component as the longest Weyl group element for that component : it sends $\alpha_k \in \Theta_i$ to $-\alpha_{\ell_i+1-k}$.

In a second time, $\widetilde{w_0^{(M_1)_\alpha}}$ operates on $\Theta_i \cup \{\alpha_i\} \cup \Theta_{i+1}$ in a similar fashion, and trivially on each component in $\bigcup_{j \neq i,i+1} \Theta_j$.
Secondly, let us assume that β is the restriction of the simple root connecting Θ_{n-1} of type A and Θ_n of type B, C or D in the Dynkin diagram of G.
$\widetilde{w_0^{(M_1)}}(\Theta_{n-1}) = \Theta_{n-1}$ (since this element simply permutes and multiply by (-1) the simple roots in Θ_{n-1}), while $\widetilde{w_0^{(M_1)}}(\Theta_n) = -\Theta_n$. Further, $\widetilde{w_0^{(M_1)_\beta}}$ acts as (-1) on all the simple roots in $\Theta_{n-1} \cup \Theta_n$.

Eventually, $\widetilde{w_0^{(M_1)_\beta}}\widetilde{w_0^{(M_1)}}$ fixes Θ_n pointwise and sends each root in Θ_{n-1} to another root in Θ_{n-1}. It also fixes pointwise $\bigcup_{j \neq n-1,n}^n \Theta_j$.

Therefore, for any α in $\Delta(P_1)$, $\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}(\Theta) = \Theta$, hence $s_\alpha \in \{w \in W | w^{-1}M_1 w = M_1\}$.

Furthermore, since the length of this element is the difference of the lengths of each element in this composition, it is clear that s_α is of minimal length in its classe in the quotient $\{w \in W | w^{-1} M_1 w = M_1\} / W^{M_1}$, hence this element is in $W(M_1)$.

Point (2)

Any element in $W(M_1)$ is a representative of minimal length in its classe in the quotient $\{w \in W | w^{-1} M_1 w = M_1\} / W^{M_1}$. The $s_\alpha = \widetilde{w_0^{(M_1)_\alpha}} \widetilde{w_0^{(M_1)}}$ described above where $\alpha \in \Delta(P_1)$ are a set of generators of $W(M_1)$. Recall from Proposition 46 that $\Delta(P_1) = \{\alpha_1, \ldots, \alpha_{d-1}, \beta_d\}$ and $\Delta_\sigma = \{\alpha_1, \ldots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D. Therefore $W(M_1) = W_\sigma \cup \{s_{\beta_d} W_\sigma\}$[6].

Point (3)

Let us denote $M_1' = M_{\Theta'}$, and $M_1 = M_\Theta$ and assume that Θ and Θ' are written as $\bigcup_{i=1}^n \Theta_i$, where, for any $i \in \{1, \ldots, n-1\}$, Θ_i is an irreducible component of type A.

Since the cuspidal data are the support of the same irreducible discrete series, by Theorem 2.9 in Bernstein et Zelevinsky 1977, there exists $w \in W^G$ such that $M_1' = w.M_1$, $\sigma' = w.\sigma$. Since M_1' is isomorphic to M_1, Θ' is isomorphic to Θ.

Therefore applying the observations made in the first part of the proof of this Proposition to M_1 and M_1', we observe Θ and Θ' share the same constraints : their components of type A are all of the same cardinal and the interval between any two of these consecutive components is of length one. Also, since Θ' is isomorphic to Θ, its last component Θ_m' is of the same type as Θ_m. Therefore $\Theta' = \Theta$.

Hence $M_1 = M_1'$. □

Remark 8. This implies that if $P_1 = M_1 U_1$ and $P_1' = M_1' U_1'$ are both standard parabolic subgroups such that their Levi subgroups satisfy the conditions of the previous Proposition, they are actually equal.

1.8.1.1.

In this section and the following the core of our argumentation relies on the form of the parameters λ ; changes on the form of these parameters are induced by actions of Weyl group elements (see for instance Example 9). In fact, the Weyl group operates on σ_λ and any Weyl group element decomposes in elementary symmetries s_{α_i} for $\alpha_i \in \Delta$. This kind of decomposition is explained in details in I.1.8 of the book Walspurger.JL 1995. If α_i is in Δ_σ, by Harish-Chandra's Theorem (Theorem 8), $s_{\alpha_i}\sigma \cong \sigma$; however recall that for $\beta_d \in \Delta(P_1)$ (see Proposition 46), we may not have $s_{\beta_d}\sigma \cong \sigma$.

6. Notice that in the context of Σ_σ of type D_d and $\Sigma(P_1)$ of type B_d or C_d : $s_{\alpha_d} = s_{\alpha_{d-1}} s_{\beta_d} s_{\alpha_{d-1}} s_{\beta_d}$

1.8.1.2. A few preliminary results for the proof of Moeglin's extended lemmas

Let us recall Casselman's square-integrability criterion as stated in Waldspurger 2003 whose proof can be found in (« Introduction to the theory of admissible representations of p-adic reductive groups »,(4.4.6)). Let $\Delta(P)$ be a set of simple roots, then ${}^{+}a_P^G*$,resp. ${}^{+}\overline{a}_P^G*$, denote the set of χ in a_M^* of the following form : $\chi = \sum_{\alpha\in\Delta(P)} x_\alpha \alpha$ with $x_\alpha > 0$,resp $x_\alpha \geq 0$. Further, denote π_P the Jacquet module of π with respect to P, and $\mathcal{E}xp$ the set of exponents of π as defined section I.3 in Waldspurger 2003.

Proposition 48 (Proposition III.1.1 in Waldspurger 2003)**.** *The following conditions are equivalent :*

1. *π is square-integrable ;*
2. *for any semi-standard parabolic subgroup $P = MU$ of G, and for any χ in $\mathcal{E}xp(\pi_P)$, $Re(\chi) \in {}^{+}a_P^G*$*
3. *for any standard parabolic subgroup $P = MU$ of G, proper and maximal, and for any χ in $\mathcal{E}xp(\pi_P)$, $Re(\chi) \in {}^{+}a_P^G*$.*

In the following two lemmas we will apply the previous Proposition as follows :

Proposition 49. *Let π_0 embed in $I_{P_1}^G(\sigma_\lambda)$. Let us write the parameter λ as a vector in the basis $\{e_i\}_i$ as $((x,y)+\underline{\lambda})$ for a linear segment (x,y), and assume $\sum_{k\in[x,y]} k \leq 0$. Then π_0 is not square-integrable.*

Proof. Indeed, if

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma((x,y)+\underline{\lambda}))$$

by Frobenius reciprocity, the character χ_λ appears as exponent of the Jacquet module of π_0 with respect to P_1. Let us write λ as

$$\sum_i x_i(e_i - e_{i+1}) + \underline{\lambda} = \sum_i y_i e_i + \underline{\lambda}$$

it is clear that, for any j, $x_j = \sum_{i=1}^{j} y_j$, and notice there is an index j' such that $x_{j'} = \sum_{k\in[x,y]} k$. Therefore, using the hypothesis of the Proposition, $x_{j'} = \sum_{k\in[x,y]} k \leq 0$. But then χ_λ does not satisfy the requirement of Proposition 48 since $x_{j'}$ is negative.

□

We will also use the following well-known result :

Theorem 50 (Renard 2010, Theorem VII.2.6)**.** *Let (π, V) be a admissible irreducible representation of G. Then (π, V) is tempered if and only if there exists a standard parabolic subgroup of G, $P = MN$, and a square integrable irreducible representation (σ, E) of M such that (π, V) is a subrepresentation of $I_P^G(\sigma)$.*

Le us repeat here Lemma 9 :

Lemma 51 (Lemma 1.8 in HEIERMANN 2011). *Let $\alpha \in \Delta_\sigma$, $s = s_\alpha$ and assume $(M_1)_\alpha$ is a standard Levi subgroup of G. The operator $J_{sP_1|P_1}$ are meromorphic functions in σ_λ for σ unitary cuspidal representation and λ a parameter in $(a_{M_1}^{(M_1)_\alpha}*)$.*

The poles of $J_{sP_1|P_1}$ are precisely the zeroes of $\mu^{(M_1)_\alpha}$. Any pole has order one and its residue is bijective. Furthermore, $J_{P_1|sP_1}J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\alpha})^{-1}$ up to a multiplicative constant.

Further by a general result concerning the μ function, it has one and only one pole on the positive real axis if and only if, for fixed σ unitary irreducible cuspidal representation, $\mu(\sigma) = 0$ (This is clear from the explicit formula given by Silberger SILBERGER 1980b).

Therefore for each $\alpha \in \Sigma_\sigma$, by definition, there will be one λ on the positive real axis such that $\mu^{(M_1)_\alpha}$ has a pole.

Lemma 52. *Let $\beta \in \Delta(P_1)$, and assume $\beta \notin \Delta_\sigma$, then the elementary intertwining operator associated to $s_\beta \in W$ is bijective at σ_λ $\forall \lambda \in a_{M_1}^*$.*

Proof. Set $s = s_\beta$ for $\beta \in \Delta(P_1)$, and $\beta \notin \Delta_\sigma$. Recall we have $J_{P_1|sP_1}J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\beta})^{-1}$ up to a multiplicative constant.

Recall $\mathcal{O}$ denotes the set of equivalence classes of representations of the form $\sigma \otimes \chi$ where χ is an unramified character of M_1.

The operator $\mu^{(M_1)_\beta} J_{P_1|s_\beta P_1}$ is regular at each unitary representation in $\mathcal{O}$ (see WALDSPURGER 2003, V.2.3), $J_{s_\beta P_1|P_1}$ is itself regular on $\mathcal{O}$, since this operator is polynomial on $X^{nr}(G)$.

By the general result mentioned after Lemma 9, the function $\mu^{(M_1)_\beta}$ has a pole at σ_λ for λ on the positive real axis, if $\mu^{(M_1)_\beta}(\sigma) = 0$. Therefore, by definition, since $\beta \notin \Delta_\sigma$, there is no pole at σ_λ.

Further, since the regular operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ are non-zero at any point, if $\mu^{(M_1)_\beta}$ does not have a pole at σ_λ, these operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ are bijective. □

A consequence of this lemma is that for any root $\beta \in \Sigma(P_1)$ which admits a reduced decomposition without elements in Δ_σ, the intertwining operators associated to s_β are everywhere bijective.

1.8.2. Extended Moeglin's Lemmas

The three next lemmas, inspired by Remark 3.2 page 154 and Lemma 5.1 in Moeglin MOEGLIN 2002 are used in our main embedding (of the irreducible generic discrete series) result.

Recall that in general $P_{\Theta'}$ is the parabolic subgroup associated to the subset $\Theta' \subset \Delta$, and $M_{\Theta'}$ contains all the roots in Θ'. Recall that we denote $\underline{\alpha_i}$ the simple root in Δ which restricts to α_i in $\Delta(P_1)$.

Definition 53. Let (M_1, σ) be the generic cuspidal support of an irreducible generic discrete series.

Let us denote $M_1 = M_\Theta$. Let us assume that $\Theta = \bigcup_{i=1}^n \Theta_i$, where, for any $i \in \{1, \dots, n-1\}$, Θ_i is an irreducible component of type A.

We say this cuspidal support satisfies the conditions *(CS)* (given in Proposition 46 and Corollary 47) if :

- Σ_σ is irreducible of rank d.
- If $\Delta(P_1) = \{\alpha_1, \dots, \alpha_{d-1}, \beta_d\}$ then $\Delta_\sigma = \{\alpha_1, \dots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D.
- For any $i \in \{1, \dots, n-1\}$, Θ_i has fixed cardinal. Furthermore, the interval between any two disjoint consecutive components Θ_i, Θ_{i+1} is of length one.

Lemma 54. *Let π_0 be a generic discrete series of a quasi-split reductive group G (of type A, B, C or D) whose cuspidal support (M_1, σ_λ) satisfies the condition* (CS) *(see the Definition 53).*

Let

$$x, y \in \mathbb{R}, k+1 = x - y \in \mathbb{N}$$

This defines the integer k.

Let us denote

$$M' = M_{\Delta - \{\underline{\alpha}_1, \dots, \underline{\alpha}_{k-1}, \underline{\alpha}_k\}}$$

Let us assume there exists $w_{M'} \in W^{M'}(M_1)$, and an irreducible generic representation τ which is the unique generic subquotient of $I_{P_1 \cap M'}^{M'}(\sigma_{\lambda_1^{M'}})$ such that

$$\pi_0 \hookrightarrow I_{P'}^G(\tau_{(x,y)}) \hookrightarrow I_{P_1}^G((w_{M'}\sigma)_{(x,y)+\lambda_1^{M'}}); \quad \lambda_1^{M'} \in a_{M_1}^{M'} \tag{1.6}$$

Let us assume y is minimal for this property.

Then τ is square integrable.

Proof. Let us first remark that in Equation 1.6 the parameter in $a_{M_1}^*$ is decomposed as $\underbrace{(x, y)}_{\text{combination of } \alpha_1, \dots, \alpha_{k-1}} + \underbrace{\lambda_1^{M'}}_{\text{combination of } \alpha_{k+1}, \dots, \beta_d}$.

Let us denote τ the generic irreducible subquotient in $I_{P_1 \cap M'}^{M'}(\sigma_{\lambda_1^{M'}})$, and let us show that τ is square integrable.

Assume on the contrary that τ is not square-integrable.

Then τ is tempered (but not square integrable) or non-tempered. Langlands' classification [Theorem 13] insures us that τ is a Langlands quotient $J(P'_L, \tau', \nu')$ for a parabolic subgroup $P'_L \supseteq P_1$ of M' or equivalently a subrepresentation in $I_{P'_L}^{M'}(\tau'_{\nu'})$, $\nu' \in \overline{((a_{M'_L}^{M'})^*)^-}$ (Equivalently $\nu'_{P'_L} \leq 0$, the inequality is strict in the non-tempered case).

This is equivalent to claim there exists an irreducible generic cuspidal representation σ', (half)-integers ℓ, m with $\ell - m + 1 \in \mathbb{N}$ and $m \leq 0$ such that :

$$\tau \hookrightarrow I_{P'_L}^{M'}(\tau'_{\nu'}) \hookrightarrow I_{P_1\cap M'}^{M'}(\sigma'((\ell,m)+\lambda_2{}^{M'})) \tag{1.7}$$

$$\sum_{k\in[\ell,m]} k \leq 0 \tag{*}$$

We have extracted the linear segment (ℓ,m) out of the segment $\lambda_1^{M'}$ and named $\lambda_2^{M'}$ what is left.

Let us justify Equation (*) : The parameter ν' reads

$$\underbrace{(\ldots,\frac{\ell+m}{2},\ldots,}_{\ell-m+1 \text{ times}}0,\ldots,0)$$

$$\nu'_{P'_L}\leq 0 \Leftrightarrow \frac{\ell+m}{2}\leq 0 \Leftrightarrow m\leq -\ell \Leftrightarrow \sum_{k\in[\ell,m]} k\leq 0$$

From Equation (1.7)

$$\pi_0 \hookrightarrow I_{P'}^{G}(\tau_{(x,y)}) \hookrightarrow I_{P_1}^{G}(\sigma'((x,y)+(\ell,m)+\lambda_2^{M'})) \tag{1.8}$$

Since π_0 also embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$, by Theorem 2.9 in Bernstein et Zelevinsky 1977 (see also Renard 2010 VI.5.4) there exists a Weyl group element w in W^G such that $w.M_1=M_1, w.\sigma'=\sigma$ and $w((x,y)+(\ell,m)+\lambda_2^{M'})=\lambda$. This means we can take w in $W(M_1)$.

But we can be more precise on this Weyl group element : from Equation 1.7, we see we can take it in $W^{M'}(M_1)$ and it leaves the leftmost part of the cuspidal support, $\sigma_{(x,y)}$, invariant, this element therefore depends on x and y. We denote this element $w_{M'}$.

Let

$$M''=M_{\Delta-\{\underline{\alpha}_q,\ldots,\underline{\beta}_d\}}$$

where $q=x-y+1+\ell-m+1$.

Now, let us consider two cases. First, let us assume $m\geq y$. If the two linear segments are unlinked and the generic subquotient in $I_{P_1\cap M''}^{M''}(\sigma'((x,y)+(\ell,m)))$ is irreducible, applying Lemma 42, we can interchange them in the above Equation (1.8) and we reach a contradiction to the Casselman Square Integrability criterion applied to the discrete series π_0 (considering its Jacquet module with respect to P_1, see Proposition 49 using $\sum_{k\in[\ell,m]} k\leq 0$).

By Proposition 38 and Remark 6, if the two linear segments are linked the irreducible generic subquotient $\tau_{L,gen}$ of

$$I_{P_1\cap M''}^{M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$

embeds in
$$I^{M''}_{P_1\cap M''}((w.w_{M'}\sigma)((\ell,y)+(x,m)))$$
(for some Weyl group element $w\in W^{M''}(M_1)$, such that $w.w_{M'}\sigma\cong w_{M'}\sigma$).

By Lemma 40 there exists an intertwining operator with non generic kernel sending $\tau_{L,gen}$ to $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$. Then by unicity of the generic piece in $I^G_{P_1}((w_{M'}\sigma)((x,y)+(\ell,m)+\lambda_2^{M'}))$, π_0 embeds in $I^G_{P''}((\tau_{L,gen})_{\lambda_2^{M'}})$.

Therefore, inducing to G, we have

$$\pi_0\hookrightarrow I^G_{P''}((\tau_{L,gen})_{\lambda_2^{M'}})\hookrightarrow I^G_{P'''}(I^{M'''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,y)+(x,m)+\lambda_2^{M'}))$$

but then since $\sum_{k\in[\ell,y]}k\leq 0$ (since $m\geq y$), we reach a contradiction to the Casselman Square Integrability criterion applied to the discrete series π_0 (considering its Jacquet module with respect to P_1).

Secondly, let us assume $m<y$. The induced representation
$$I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$
is reducible only if $\ell\in]x,y-1]$. Then using Proposition 38 and Remark 6, we know that the irreducible generic subquotient $\tau_{L,gen}$ of
$$I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$
should embed in
$$I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,m)+(\ell,y)))$$
(or only $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,m)))$ if $\ell=y-1$).

Applying Lemma 42, we also know that it embeds in $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,y)+(x,m)))$ (we can interchange the order of the two unlinked segments (ℓ,y) and (x,m)). Then, using Lemma 40 and unicity of the generic irreducible piece as above, we embed π_0 in $I^G_{P''}((\tau_{L,gen})_{\lambda_2^{M'}})\hookrightarrow I^G_{P_1}((w_{M'}\sigma)((x,y)+(\ell,m)+\lambda_2^{M'}))$.

But π_0 does not embed in $I^G_{P_1}((w_{M'}\sigma)((x,m)+(\ell,y)+\lambda_2^{M'})))$ since y is minimal for such (embedding) property.

Therefore, $\tau_{L,gen}$ rather embeds in the quotient $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,m)+(x,y)))$ of $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$.

Then π_0 embed in

$$I^G_{P''}((\tau_{L,gen})_{\lambda_2^{M'}})\hookrightarrow I^G_{P''}(I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,m)+(x,y))))_{\lambda_2^{M'}}=I^G_{P_1}((w_{M'}\sigma)((\ell,m)+(x,y)+\lambda_2^{M'}))$$

Since $\sum_{k\in[\ell,m]}k\leq 0$, using Proposition 49, we reach a contradiction. $\square$

Lemma 55. *Let π_0 be a generic discrete series of G whose cuspidal support satisfies the conditions* CS *(see the Definition 53). Let a,a_- be two consecutive jumps in the set of Jumps of π_0.*

Let us assume there exists an irreducible representation π' of a standard Levi $M' = M_{\Delta - \left\{\underline{\alpha}_1, \dots, \underline{\alpha}_{a-a_-}\right\}}$ such that

$$\pi_0 \hookrightarrow I_{P'}^G(\pi'_{(a,a_-+1)}) \hookrightarrow I_{P_1}^G(\sigma_{(a,a_-+1)+\lambda}). \tag{1.9}$$

Then there exists a generic discrete series π of $M'' = M_{\Delta - \left\{\underline{\alpha_{a+a_-+1}}\right\}}$ such that :

π_0 embeds in $I_{P''}^G((\pi)_{s\widetilde{\alpha}_{a+a_-}}) \hookrightarrow I_{P_1}^G(\sigma((a, -a_-) + (\underline{n})))$ with $s = \frac{a-a_-}{2}$ and $(\underline{n})$ a residual segment.

We split the proof in two steps :

Step A

We first need to show that π' is necessarily tempered following the argumentation given in Moeglin 2002.

Assume on the contrary that π' is not tempered. Langlands' classification [Theorem 13] insures us that π' is a subrepresentation in $I_{P_L}^{M'}(\tau_\nu)$, for a parabolic standard subgroup $P_L \supseteq P_1$ and

$$\nu \in ((a_L^{M'})^*)^-$$

This is equivalent to claim there exists x, y with $x - y + 1 \in \mathbb{N}$, and $y \leq 0$, a Levi subgroup

$$L = M_{\Delta - \left\{\underline{\alpha}_1, \dots, \underline{\alpha}_{a-a_-}\right\} \cup \left\{\underline{\alpha}_{x-y}\right\}}$$

a unitary cuspidal representation $w_{M'}\sigma$ in the $W(M_1)^{M'}$ group orbit of σ, and the element $\lambda \in (a_{M_1}^{M'})^*$ decomposes as $(x, y) + \lambda_1^{M'}$ such that :

$$\pi' = I_{P_L}^{M'}(\tau_\nu) \hookrightarrow I_{P_1 \cap M'}^{M'}((w_{M'}\sigma)((x, y) + \lambda_1^{M'}))$$

$$\sum_{k \in [x,y]} k < 0 \tag{*}$$

The first equality in the first equation is due to the Standard module conjecture since π' is generic. The second equation (*) results from the following sequences of equivalences : $\nu <_{P_L} 0 \Leftrightarrow \frac{x+y}{2} < 0 \Leftrightarrow y < -x \Leftrightarrow \sum_{k \in [x,y]} k < 0$.

The element $w_{M'}$ in $W(M_1)^{M'}$ leaves the leftmost part, $\sigma_{(a,a_-+1)}$, invariant.

Then from Equation (1.9) and inducing to G :

$$\pi_0 \hookrightarrow I_{P_1}^G((w_{M'}\sigma)((a, a_- + 1) + (x, y) + \lambda_1^{M'}))$$

We can change $(a, a_-+1)(x, y)$ to $(x, y)(a, a_-+1)$ if and only if the two segments $(a, \dots, a_- + 1)$ and $(x, \dots, y)$ are unlinked (see the Lemma 42). As $y \leq 0$, this condition is equivalent to $x \notin]a, a_-]$.

If we can change, since $\sum_{k\in[x,y]} k < 0$, we get by Proposition 49 a contradiction to the square integrability of π_0.

Assume therefore we cannot change, then the two segments are linked by Proposition 37. Let $M''' = M_{\Delta - \{\underline{\alpha}_q, \ldots, \underline{\beta}_d\}}$ where $q = a - a_- + x - y + 1$.

The induced representation

$$I_{P_1 \cap M'''}^{M'''}((w_{M'}\sigma)((a, \ldots, a_- + 1) + (x, \ldots, y)))$$

has a generic submodule which is :

$$Z^{M'''}(P_1, w_L.w_{M'}\sigma, (a, \ldots, y)(x, \ldots, a_- + 1))$$

(for some Weyl group element w_L such that $w_L.w_{M'}\sigma \cong w_{M'}\sigma$)

We twist these by the character $\lambda_1^{M'}$ central for M'''.

and therefore, by unicity of the irreducible generic piece :

$$\pi_0 \hookrightarrow I_{P'''}^G(Z^{M'''}(P_1, w_{M'}\sigma, (a, \ldots, y)(x, \ldots, a_- + 1))_{\lambda_1^{M'}})$$
$$\hookrightarrow I_{P'''}^G(I_{P_1 \cap M'''}^{M'''}((w_{M'}\sigma)((a, \ldots, a_- + 1) + (x, \ldots, y)))_{\lambda_1^{M'}}) =$$
$$I_{P_1}^G((w_{M'}\sigma)((a, \ldots, y) + (x, \ldots, a_- + 1) + \lambda_1^{M'})$$

Let $Q' = L'U'$, we rewrite this as :

$$\pi_0 \hookrightarrow I_{Q'}^G(Z^{L'}(P_1, w'_L.w_{M'}\sigma, (a, \ldots, y)(\lambda_2^{M'}))) \hookrightarrow I_{P_1}^G((w'_L.w_{M'}\sigma)((a, \ldots, y) + \lambda_2^{M'}))$$
$$:= I_{P_1}^G((w_{M'}\sigma)((a, \ldots, y) + \lambda_2^{M'}))$$

for some Weyl group element w'_L such that $w'_L.w_{M'}\sigma \cong w_{M'}\sigma$.

Further, we have $y < -a_-$ since y is negative, $x \geq a_-$ and $\sum_{k\in[x,y]} k < 0$. In this context, the above Lemma 54 claims there exists $y' \leq y$:

$$\pi_0 \hookrightarrow I_{P_1}^G((w_{M'}\sigma)((a, \ldots, y') + \lambda_3^{M'}))$$

And then the unique irreducible generic subquotient π'_0 of $I_{P_1 \cap N'}^{N'}(\sigma_{\lambda_3^{M'}})$ is square-integrable, or equivalently $\sigma_{\lambda_3^{M'}}$ is a residual point for $\mu^{N'}$ (The type is given by $\Sigma_\sigma^{N'}$). Further, $\sigma_{(a,\ldots,y')+\lambda_3^{M'}}$ is a residual point for μ^G (type given by Σ_σ), corresponding to the generic discrete series π_0.

Then the *set of Jumps* of the residual segment associated to π_0 contains the *set of Jumps* of the residual segment associated to π'_0 and two more elements a and $-y'$ but then $a > -y' > a_-$ and this contradicts the fact that a and a_- are two consecutive jumps.

We have shown that π' is necessarily tempered.

Step B

Let $(\underline{n}_{\pi_0})$ be the residual segment canonically associated to a generic discrete series π_0. Let us now denote a_{i+1} the greatest integer smaller than a_i in the *set of Jumps* of $(\underline{n}_{\pi_0})$. Therefore, the half-integers, a_i and a_{i+1} satisfy the conditions of this lemma.

As the representation π' is tempered, by Theorem 50, there exists a standard parabolic subgroup $P_\#$ of M' and a discrete series τ' such that $\pi' \hookrightarrow I_{P_\#}^{M'}(\tau')$.

Again, as an irreducible generic discrete series representation of a non necessarily maximal Levi subgroup, using the result of Heiermann-Opdam (Proposition 12), there exists an irreducible cuspidal representation σ' and a standard parabolic $P_{1,\#}$ of $M_\#$ such that τ' embeds in $I_{P_{1,\#}}^{M_\#}(\sigma'((\frac{a-a_--1}{2}, -\frac{a-a_--1}{2}) + \bigoplus_j(a_j, -a_j) + (\underline{n}_{\pi_0''})))$, where $(\underline{n}_{\pi_0''})$ is a residual segment corresponding to an irreducible generic discrete series π_0'' and $(\frac{a-a_--1}{2}, -\frac{a-a_--1}{2})$ along with $(a_j, -a_j)$'s are linear residual segments for (half)-integers a_j.

Clearly, the point $(\frac{a-a_--1}{2}, \ldots, -\frac{a-a_--1}{2}) + \bigoplus_j(a_j, -a_j) + (\underline{n}_{\pi_0''})$ is in $\overline{a_{M_1}^{M_\# *}}^+$.
Then

$$\pi' \hookrightarrow I_{P_{1,\#}U_\#}^{M'}(\sigma'((\frac{a-a_--1}{2}, \ldots, -\frac{a-a_--1}{2}) + \bigoplus_j(a_j, \ldots, -a_j) + (\underline{n}_{\pi_0''}))) \tag{1.10}$$

Since $P_{1,\#}U_\#$ is standard in P' which is standard in G, there exists a standard parabolic subgroup P_1' in G, such that, when inducing Equation 1.10, we obtain :

$$\pi_0 \hookrightarrow I_{P'}^G(\pi'_{(a,\ldots,a_-+1)}) \hookrightarrow I_{P_1'}^G(\sigma'_{(a,\ldots,a_-+1)+\bigoplus_j(a_j,\ldots,-a_j)+(\underline{n}_{\pi_0''})}) \tag{1.11}$$

Let us denote $(a, \ldots, a_- + 1) + \bigoplus_j(a_j, \ldots, -a_j) + (\underline{n}_{\pi_0''}) := \lambda'$.

Since π_0 also embeds as a subrepresentation in $I_{P_1}^G(\sigma_{(a,\ldots,a_-+1)+\lambda})$, by Theorem 2.9 in Bernstein et Zelevinsky 1977 (see also Renard 2010 VI.5.4) there exists a Weyl group element w in W^G such that $w.M_1 = M_1', w.\sigma = \sigma'$ and $w((a, a_- + 1) + \lambda) = \lambda'$.

Since Σ_σ is irreducible and M_1' is standard, we have by Point (3) in Corollary 47 that $M_1' = M_1$, and we can take w in $W(M_1)$. Further since P_1 and P_1' are standard parabolic subgroups of G, and Σ_σ is irreducible they are actually equal (see Remark 8).

Now, by Point (2) in Corollary 47 any element in $W(M_1)$ is either in W_σ or decomposes in elementary symmetries in W_σ and $s_{\beta_d}W_\sigma$ and :

$$\sigma' = w\sigma = \begin{cases} \sigma \text{ if } w \in W_\sigma \\ \text{Else } s_{\beta_d}\sigma \end{cases}$$

Let us assume we are in the context where $\sigma' = s_{\beta_d}\sigma \ncong \sigma$. As explained in the first part of Section 1.8.1, this happens if Σ_σ is of type D.

Let us apply the bijective operator (see Lemma 52) from $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(s_{\beta_d}\sigma)_{\lambda'})$ to $I_{\overline{P_1}\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ and then the bijective map $t(s_{\beta_d})$ (the definition of the map $t(g)$ has been given in the proof of Proposition 16) to $I_{s_{\beta_d}(\overline{P_1}\cap(M_1)_{\beta_d})}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}}\lambda') = I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}}\lambda')$.

As explained in Remark 7, $s_{\beta_d}\lambda' = \lambda'$ since λ' is a residual point of type D.

Therefore, we have a bijective map from $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(s_{\beta_d}\sigma)_{\lambda'})$ to $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{\lambda'})$.

The induction of this bijective map gives a bijective map from $I_{P_1'}^G(\sigma'_{(a,\ldots,a_-+1)+\bigoplus_j(a_j,\ldots,-a_j)+(\underline{n}_{\pi_0''})})$ to $I_{P_1}^G(\sigma_{(a,\ldots,a_-+1)+\bigoplus_j(a_j,\ldots,-a_j)+(\underline{n}_{\pi_0''})})$.

Therefore we may write Equation 1.11 as :

$$\pi_0 \hookrightarrow I_{P'}^G(\pi'_{(a,\ldots,a_-+1)}) \hookrightarrow I_{P_1}^G(\sigma_{(a,\ldots,a_-+1)+\bigoplus_j(a_j,\ldots,-a_j)+(\underline{n}_{\pi_0''})}) \tag{1.12}$$

Let us set $a = a_i$, $a_- = a_{i+1}$ for a_i, a_{i+1} two consecutive elements in the set of Jumps of $(\underline{n}_{\pi_0})$.

Therefore, $(a_i,\ldots,\ldots,a_{i+1}+1)\bigoplus_j(a_j,\ldots,-a_j)+(\underline{n}_{\pi_0''})$ is in the Weyl group orbit of the residual segment associated to π_0 : $(\underline{n}_{\pi_0})$.

Let us show that $(a_i,\ldots,a_{i+1}+1)(a_{i+1},\ldots,-a_{i+1})(\underline{n}^i)$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

One notices that in the tuple $\underline{n}_{\pi_0}$ of the residual segment $(\underline{n}_{\pi_0})$ the following relations are satisfied :

$$n_{a_i} = n_{a_{i+1}} - 1 \tag{1.13}$$

$$n_i = n_{i-1} - 1 \text{ or } n_i = n_{i-1},\ \forall i > 0 \tag{1.14}$$

Therefore, when we withdraw $(a_i,\ldots,a_{i+1}+1)$ from this residual segment, we obtain a segment $(\underline{n}')$ which cannot be a residual segment since $n'_{a_{i+1}} = n'_{a_{i+1}+1}+2$ for $i\neq 1$; or if $i=1$, $n'_{a_2}=2$ but a_2 is now the greatest element in the set of Jumps associated to the segment $(\underline{n}')$, so we should have $n'_{a_2}=1$.

Therefore, to obtain a residual point (residual segment $(\underline{n}_{\pi_0''})$), we need to remove twice a_{i+1}.

Then, for any $0<j<a_{i+1}$, if we remove twice j, $n'_j = n_j - 2$ and, for all i, the relations $n'_j = n'_{j-1}-1$ or $n'_j = n'_{j-1}$ are still satisfied. As we also remove one zero, we have for $j=0$, $n'_0 = n_0 - 1$ which is compatible with removing twice $j=1$.

The residual segment left, thus obtained, will be denoted $(\underline{n}^i)$. We have shown that $(a_i,\ldots,a_{i+1}+1)(a_{i+1},\ldots,-a_{i+1})(\underline{n}^i)$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

Since $(\underline{n}^i)$ is a residual segment, from the conditions detailed in Equations 1.13 and 1.14 (see also Remark 4 in Section 1.5.2) no symmetrical linear residual segment $(a_k,-a_k)$ can be extracted from $(\underline{n}^i)$ to obtain another residual segment $(\underline{n}_{\pi_0''})$ such that $(a_i,\ldots,a_{i+1}+1)(a_{i+1},\ldots,-a_{i+1})(a_k,-a_k)(\underline{n}_{\pi_0''})$ is in the W_σ orbit of $(\underline{n}_{\pi_0})$.

So $(\underline{n}_{\pi_0''}) = (\underline{n}^i)$ and

$$\pi'_{(a,a_-+1)} \hookrightarrow I_{P_1}^{M'}(\sigma((a_i, a_{i+1}+1) + (a_{i+1}, -a_{i+1}) + (\underline{n}^i)))$$

Eventually, using induction in stages Equation 1.10 rewrites :

$$\pi_0 \hookrightarrow I_{P_1}^{G}(\sigma((a_i, a_{i+1}+1) + (a_{i+1}, -a_{i+1}) + (\underline{n}^i)) = \Theta$$

and since the two segments $(a_i, \ldots, a_{i+1}+1)$ and $(a_{i+1}, \ldots, -a_{i+1})$ are linked, we can take their union and deduce there exists an irreducible generic essentially square integrable representation π_i of a Levi subgroup M^i (this notation overlaps with another, but this Levi M^i is not the same as the one used in Proposition 24 and defined in Section 1.20) in P^i which once induced embeds as a subrepresentation in Θ and therefore by multiplicity one of the irreducible generic piece, π_0, we have :

$$\pi_0 \hookrightarrow I_{P^i}^{G}(\pi_i) \hookrightarrow I_{P_1}^{G}(\sigma((a_i, -a_{i+1}) + (\underline{n}^i)))$$

Proposition 56. *Let $(\underline{n}_{\pi_0})$ be a residual segment associated to an irreducible generic discrete series π_0 of G whose cuspidal support satisfies the conditions* CS *(see the Definition 53).*

Let $a_1 > a_2 > \ldots > a_n$ be Jumps *of this residual segment. Let $P_1 = M_1U_1$ be a standard parabolic subgroup, σ be a unitary irreducible cuspidal representation of M_1 such that $\pi_0 \hookrightarrow I_{P_1}^{G}(\sigma(\underline{n}_{\pi_0}))$.*

For any i, there exists a standard parabolic subgroup $P^i \supset P_1$ with Levi subgroup M^i (this Levi M^i is not the same as the one used in Proposition 24 and defined in Section 1.20), residual segment $(\underline{n}^i)$ and an irreducible generic essentially square-integrable representation $\pi_{a_i} = Z^{M^i}(P_1, \sigma, (a_i, -a_{i+1})(\underline{n}^i))$ such that π_0 embeds as a subrepresentation in

$$I_{P^i}^{G}(\pi_{a_i}) \hookrightarrow I_{P_1}^{G}((\sigma((a_i, -a_{i+1}) + (\underline{n}^i))))$$

Proof. By the result of Heiermann-Opdam [Proposition 12] and Lemma 34, to any residual segment $(\underline{n}_{\pi_0})$ we associate the unique irreducible generic discrete series subquotient in $I_{P_1}^{G}(\sigma(\underline{n}_{\pi_0}))$.

Then as explained in the Subsection 1.5.2 this residual segment defines uniquely *Jumps* : $a_1 > a_2 > \ldots > a_n$.

Start with the two elements $a_1 = \ell + m$ and $a_2 = \ell - 1$ and consider the following induced representation :

$$\begin{aligned} I_{P_1}^{G}&(\sigma((\ell+m, a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})) \\ &= I_{P}^{G}(I_{P_1\cap M}^{M}(\sigma((\ell+m, a_2=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})) \quad (1.15) \end{aligned}$$

Let us denote $\nu := (\ell+m, a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 0^{n_0})$.

The induced representation $I_{P_1\cap M}^M(\sigma(\ell+m, a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 0^{n_0}) := I_{P_1\cap M}^M(\sigma_\nu)$ is a generic induced module.

The form of ν implies σ_ν is not necessarily a residual point for μ^M. Indeed, the first linear residual segment $(\ell+m, a_2+1=\ell)$ is certainly a residual segment (of type A), but the second not necessarily.

Let π be the unique irreducible generic subquotient of $I_{P_1\cap M}^M(\sigma_\nu)$ (which exists by Rodier's Theorem). We have : $\pi \leq I_{P_1\cap M}^M(\sigma_\nu)$ and $I_P^G(\pi) \leq I_P^G(I_{P_1\cap M}^M(\sigma_\nu)) := I_{P_1}^G(\sigma_\lambda)$.

Assume $I_P^G(\pi)$ has an irreducible generic subquotient π_0' different from π_0, then π_0' and π_0 would be two generic irreducible subquotients in $I_{P_1}^G(\sigma_\lambda)$ contradicting Rodier's theorem. Hence $\pi_0 \leq I_P^G(\pi)$.

Further, since π_0 embeds as a subrepresentation in

$$I_P^G(I_{P_1\cap M}^M(\sigma((\ell+m, a_2+1=\ell)+(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 0^{n_0})) := I_{P_1}^G(\sigma_\lambda)$$

it also has to embed as a subrepresentation in $I_P^G(\pi)$.

Therefore applying Lemma 55, we conclude there exists a residual segment $(\underline{n}^1)$ an essentially square integrable representation π_1 such that π_0 embeds as a subrepresentation in

$$I_{P^1}^G(\pi_{a_1}) \hookrightarrow I_{P_1}^G((\sigma((a_1, -a_2)+(\underline{n}^1)))$$

Let us consider now the elements $a_2=\ell-1$ and a_3. As in the proof of Lemma 42, since the segments $(a_1, \ell-1)$ and $(\ell-1)$ are unlinked, we apply a composite map from the induced representation $I_{P_1\cap M'}^{M'}(\sigma((a_1,\ell-1)+(\ell-1)+\dots 0^{n_0}))$ to $I_{P_1\cap M'}^{M'}(\sigma((\ell-1)+(a_1,\ell-1))+\dots 0^{n_0}))$. We can interchange the two segments and as in the proof of Lemma 42, applying this intertwining map and inducing to G preserves the unique irreducible generic subrepresentation of $I_{P_1}^G(\sigma_\lambda)$.

We repeat this argument with

$$I_{P_1\cap M''}^{M''}(\sigma((\ell-1)+(a_1,\ell-2)+(\ell-2)+\dots 0^{n_0})) \text{ and } \quad I_{P_1\cap M''}^{M''}(\sigma(\ell-1)+(\ell-2)+(a_1,\ell-2)+\dots 0^{n_0}))$$

and further repeat it with all exponents til a_3+1.

Eventually, the unique irreducible subrepresentation π_0 appears as a subrepresentation in $I_{P_1}^G(\sigma((a_2,a_3+1)+(a_1,a_3+1)+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0})$.

$$\pi_0 \hookrightarrow I_{P'^2}^G(I_{P_1\cap M'^2}^{M'^2}(\sigma(a_2,a_3+1)+(a_1,a_3+1)+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0})) \\ := I_{P'^2}^G(I_{P_1\cap M'^2}^{M'^2}((w\sigma)_{w\nu}))$$

where $w \in W_\sigma$.

Let π be the unique irreducible generic subquotient of $I_{P_1\cap M'^2}^{M'^2}(\sigma_{w\nu})$ (which exists by Rodier's Theorem). We have : $\pi \leq I^{M'^2}_{P_1 \cap M'^2}(\sigma_{w\nu})$ and

$$I^G_{P'^2}(\pi) \leq I^G_{P'^2}(I^{M'^2}_{P_1 \cap M'^2}(\sigma_{w\nu})) := I^G_{P_1}(\sigma_{w\lambda})$$

Assume $I^G_{P'^2}(\pi)$ has an irreducible generic subquotient π'_0 different from π_0, then π'_0 and π_0 would be two generic irreducible subquotients in $I^G_{P_1}((w\sigma)_{w\lambda})$ contradicting Rodier's theorem. Hence $\pi_0 \leq I^G_{P'^2}(\pi)$.

Further, since π_0 embeds as a subrepresentation in

$$I^G_{P'^2}(I^{M'^2}_{P_1 \cap M'^2}(\sigma((a_2, a_3+1)+(a_1, a_3+1)+(\ell-2)^{n_{\ell-2}-2} \ldots (a_3+1)^{n_{a_3+1}-2} \ldots 1^{n_1} 0^{n_0}) := I^G_{P_1}(\sigma_{w\lambda})$$

it also embeds as a subrepresentation in $I^G_{P'^2}(\pi)$.

Therefore applying Lemma 55, we conclude there exists a residual segment $(\underline{n}^2)$ and an essentially square- integrable representation $\pi_{a_2} = Z^{M^2}(P_1 \cap M^2, \sigma, (a_2, -a_3)(\underline{n}^2))$ such that π_0 embeds as a subrepresentation in $I^G_{P^2}(\pi_{a_2}) \hookrightarrow I^G_{P_1}(\sigma((a_2, -a_3) + (\underline{n}^2))$.

Similarly, for any two consecutive elements in the *set of Jumps*, a_i and a_{i+1}, the same argumentation (i.e first embedding π_0 as a subrepresentation in $I^G_{P'^i}(\pi)$ using intertwining operators, and conclude with Lemma 79) yields the embedding :

$$\pi_0 \hookrightarrow I^G_{P^i}(\pi_{a_i}) \hookrightarrow I^G_{P^i}(I^{P^i}_{P_1 \cap M^i}(\sigma((a_i, -a_{i+1}) + (\underline{n}^i)))$$

for an irreducible generic essentially square-integrable representation

$$\pi_{a_i} = Z^{M^i}(P_1 \cap M^i, \sigma, (a_i, -a_{i+1})(\underline{n}^i))$$

of the Levi subgroup M^i.

□

1.8.3.

We present here the proof of the generalized injectivity conjecture in the case of a standard module induced from a maximal parabolic $P = MU$. Then, the roots in Lie(M) are all the roots in Δ but α. We first present the proof in case α is not an extremal root in the Dynkin diagram of G, and secondly when it is an extremal root.

The context is the one of the previous Subsection : G is a quasi-split reductive group, of type A, B, C or D and Σ_σ is irreducible.

Proposition 57. *Let π_0 be an irreducible generic representation of a quasi-split reductive group G of type A, B, C or D which embeds as a subquotient in the standard module $I^G_P(\tau_{s\tilde{\alpha}})$, with $P = MU$ a maximal parabolic subgroup and τ discrete series of M.*

Let σ_ν be in the cuspidal support of the generic discrete series representation τ of the maximal Levi subgroup M and we take $s\tilde{\alpha}$ in $\overline{(a^*_M)^+}$, such that $I^G_P(\tau_{s\tilde{\alpha}}) \hookrightarrow I^G_{P_1}(\sigma_{\nu+s\tilde{\alpha}})$ and denote $\lambda = \nu + s\tilde{\alpha}$ in $\overline{a^M_{M_1}}^{+*}$.

Let us assume that the cuspidal support of τ satisfies the conditions CS *(see the Definition 53).*

Let us assume that α is not an extremal simple root on the Dynkin diagram of Σ.

Let us assume σ_λ is a residual point for μ^G. This is equivalent to say that the induced representation $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient. Then, this unique irreducible generic subquotient, π_0, which is discrete series embeds as a submodule in $I_{P_1}^G(\sigma_\lambda)$ and therefore in the standard module $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof. First, notice that if $s = 0$, the induced module $I_P^G(\tau_{s\tilde{\alpha}})$ is unitary hence any irreducible subquotient is a subrepresentation ; in the rest of the proof we can therefore assume $s\tilde{\alpha}$ in $(a_M^*)^+$.

Let us denote π_0 the irreducible generic discrete series representation which appears as subquotient in a standard module $I_P^G(\tau_{s\tilde{\alpha}})$ induced from a maximal parabolic subgroup P of G. We are in the context of the subsection 1.5.3, and therefore we can write $\lambda := (\mathfrak{a}, \ldots, \mathfrak{b})(\underline{n})$, for some (half)- integers $\mathfrak{a} > \mathfrak{b}$, and residual segment $(\underline{n})$. In this context, as we denote $s\tilde{\alpha}$ the Langlands parameter twisting the discrete series τ, then $s = s_{\mathfrak{b}} = \frac{\mathfrak{a}+\mathfrak{b}}{2}$.

Notice that since σ_λ is in the W_σ orbit of a dominant residual point whose parameter corresponds to a residual segment of type B, C or D, $\mathfrak{a}$ and $\mathfrak{b}$ are not only reals but *(half)- integers*.

Let $\sigma'_{\lambda'}$ be the dominant residual point.

By Proposition 12, there exists a parabolic subgroup P' such that π_0 embeds as a subrepresentation in the induced module $I_{P'}^G(\sigma'_{\lambda'})$, for $\sigma'_{\lambda'}$ a dominant residual point for P'.

Let $(w\sigma)_{w\lambda}$ be the dominant (for P_1) residual point in the W_σ-orbit of σ_λ, then (using Theorem 2.9 in Bernstein et Zelevinsky 1977 or Theorem VI.5.4 in Renard 2010) π_0 is the unique irreducible generic subquotient in $I_{P_1}^G((w\sigma)_{w\lambda})$, and Proposition 16 gives us that these two $(I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G((w\sigma)_{w\lambda}))$ are isomorphic.

The point $(w\sigma)_{w\lambda}$ is a dominant residual point with respect to $P_1 : w\lambda \in \overline{a_{M_1}^*}^+$ and there is a unique element in the orbit of the Weyl group W_σ of a residual point which is dominant and is explicitely given by a residual segment using the correspondence of the Subsection 2.5.1. We denote $w\lambda := (\underline{n}_{\pi_0})$ this residual segment. Since $w \in W_\sigma$, $(w\sigma)_{w\lambda} \cong \sigma_{w\lambda}$.

Hence

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\underline{n}_{\pi_0}))$$

Since $\mathfrak{a} > \mathfrak{b}$, and $(\underline{n}_{\pi_0})$ is a residual segment, it is clear that $\mathfrak{a}$ is a jump. [Indeed, if you extract a linear residual segment $(\mathfrak{a}, \ldots, \mathfrak{b})$ such that $\mathfrak{a} > \mathfrak{b}$ from $(\underline{n}_{\pi_0})$ such that what remains is a residual segment, then $\mathfrak{a} = a$ has to be in the *set of Jumps* of the residual segment $(\underline{n}_{\pi_0})$ as defined in Section 2.5.2]. Let us denote a_- the greatest integer smaller than a in the *set of Jumps*. Therefore, the (half)-integers, a and a_- satisfy the conditions of Proposition 56. We will further show in the next paragraph that $\mathfrak{b} \geq -a_-$. Let $P_\flat = P_{\Delta - \{\alpha_{a+a_-+1}\}}$ be a maximal parabolic subgroup, with Levi subgroup $M_\flat$, which contains P_1.

Let $\pi_a = Z^{M_\flat}(P_1, \sigma, w_{a_-}\lambda)$, for $w_{a_-} \in W_\sigma$ be the generic essentially square integrable representation with cuspidal support $(\sigma((a, -a_-)(\underline{n_{-a_-}}))$ associated to the residual segment $((a, -a_-) + (\underline{n_{-a_-}}))$ (in the Weyl group orbit of $(\underline{n}_{\pi_0})$).

It is some discrete series twisted by the Langlands parameter $s_{-a_-}\widetilde{\alpha_{a+a_-+1}}$ with $s_{-a_-} = \frac{a-a_-}{2}$.

By the Proposition 56 we can write

$$\pi_0 \hookrightarrow I_{P_\flat}^G(\pi_a) \hookrightarrow I_{P_1}^G(\sigma((a, -a_-)(\underline{n_{-a_-}}))) \tag{1.16}$$

Here, we need to justify that given a, for any $\flat$ we have : $\flat \geq -a_-$.

Consider again the residual segment $(\underline{n}_{\pi_0})$, and observe that by definition the sequence $(a, \ldots, -a_-)$ is the longest linear segment with greatest (half)-integer a that one can withdraw from $(\underline{n}_{\pi_0})$ such that the remaining segment $(\underline{n_{-a_-}})$ is a residual segment of the same type and $(a, \ldots, -a_-)(\underline{n_{-a_-}})$ is in the Weyl group orbit of $(\underline{n}_{\pi_0})$.

Further, this is true for any couple (a, a_-) of elements in the *set of Jumps* associated to the residual segment $(\underline{n}_{\pi_0})$. It is therefore clear that given a and a_- such that $s_{-a_-} = \frac{a-a_-}{2} > 0$ is the smallest positive (half)-integers as possible, we have $s_\flat = \frac{a+b}{2} \geq s_{-a_-} = \frac{a-a_-}{2}$ and $\flat$ is necessarily greater or equal to $-a_-$.

Once this embedding given, using Lemma 39, there exists an intertwining operator with non-generic kernel from the induced module $I_{P_1}^G(\sigma((a, -a_-)(\underline{n_{-a_-}})))$ given in Equation (1.16) to any other induced module from the cuspidal support $\sigma(a, \flat, \underline{n_\flat})$ with $\flat \geq -a_-$.

Therefore

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(a, \flat, \underline{n_\flat})) = I_{P_1}^G(\sigma_\lambda)$$

By multiplicity one, it will also embed as a subrepresentation in the standard module $I_P^G(Z^M(P_1, \sigma, \lambda))$.

By the above notice, if π_0 appears as a submodule in the standard module

$$I_{P_\flat}^G(Z^{M_\flat}(P_1, \sigma, w_{a_-}\lambda))$$

with Langlands parameter $s_{a_-}\widetilde{\alpha_{a+a_-+1}}$, it also appears as a submodule in any standard module $I_P^G(Z^M(P_1, \sigma, (a, \flat, \underline{n_\flat}))$ with Langlands' parameter $s_\flat\tilde{\alpha} \geq s_{-a_-}\widetilde{\alpha_{a+a_-+1}}$ for the order defined in Lemma 31 as soon as $Z^M(P_1, \sigma, (a, \flat, \underline{n_\flat}))$ has equivalent cuspidal support.

□

1.8.3.1. The case of Σ_σ^M irreducible

Proposition 58. *Let π_0 be an irreducible generic discrete series of G with cuspidal support (M_1, σ) and let us assume Σ_σ is irreducible. Let M be a standard maximal Levi subgroup such that Σ_σ^M is irreducible.*

Then, π_0 embeds as a subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where τ is an irreducible generic discrete series of M.

Proof. Assume Σ_σ is irreducible of rank d, let $\Delta_\sigma := \{\alpha_1, \ldots, \alpha_d\}$ be the basis of Σ_σ (following our choice of basis for the root system of G) and let us denote $\mathcal{T}$ its type.

We consider maximal standard Levi subgroups of G, $M \supset M_1$, such that the root system Σ_σ^M is irreducible. Typically if $M = M_{\Delta - \{\underline{\beta_d}\}}$.

Now, in our setting, σ_ν is a residual point for μ^M. It is in the cuspidal support of the generic discrete series τ if and only if (applying Proposition 24) : $rk(\Sigma_\sigma^M) = d-1$.

Let us denote $(\nu_2, \ldots, \nu_d)$ the residual segment corresponding to the irreducible generic discrete series τ of M.

If $(\nu_2, \ldots, \nu_d)$ is a residual segment of type A to obtain a residual segment $(\nu_1, \nu_2, \ldots, \nu_d)$ of rank d and type :

— D : we need $\nu_d = 0$ and $\nu_1 = \nu_2 + 1$

— B : we need $\nu_d = 1$ and $\nu_1 = \nu_2 + 1$

— C : we need $\nu_d = 1/2$ and $\nu_1 = \nu_2 + 1$

If $(\nu_2, \ldots, \nu_d)$ is a residual segment of type $\mathcal{T}$ (B, C, D) we need $\nu_1 = \nu_2 + 1$ to obtain a residual segment of type $\mathcal{T}$ and rank d.

In all these cases, the twist $s\tilde{\alpha}$ corresponds on the cuspidal support to add one element on the left to the residual segment $(\nu_2, \ldots, \nu_d)$; then the segment $(\nu_1, \nu_2, \ldots, \nu_d) := (\lambda_1, \lambda_2, \ldots, \lambda_d)$ is a residual segment :

$$\pi_0 \leq I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$$

This is equivalent to say σ_λ is a *dominant* residual point and therefore, by Lemma 34, π_0 embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and therefore in $I_P^G(\tau_{s\tilde{\alpha}})$ by multiplicity one of the generic piece in the standard module. □

1.8.4. Non necessarily maximal parabolic subgroups

In the course of the main theorem in this section, we will need the following result :

Lemma 59. *Let $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t$ be t unlinked linear segments with $\mathcal{S}_i = (a_i, \ldots, b_i)$ for any i. If $(a_1, \ldots, b_1)(a_2, \ldots, b_2)\ldots(a_t, \ldots, b_t)(\underline{n})$ is a residual segment $(\underline{n'})$; then at least one segment $(a_i, \ldots, b_i)$ merges with $(\underline{n})$ to form a residual segment $(\underline{n''})$.*

Proof. Consider the case of t unlinked segments, with at least one disjoint from the others, we aim to prove that this segment can be inserted into $(\underline{n})$ independently of the others to obtain a residual segment. For each such (disjoint from the others) segment $(a_i, \ldots, b_i)$, inserted, the following conditions are satisfied :

$$\begin{cases} n'_{a_i+1} = n_{a_i+1} = n'_{a_i} - 1 = n_{a_i} + 1 - 1 \\ n'_{b_i} = n_{b_i} + 1 = n_{b_i-1} - 1 + 1 = n_{b_i-1} = n'_{b_i-1} \end{cases} \tag{1.17}$$

The relations $n'_{a_i+1} = n_{a_i+1}$ and $n'_{b_i-1} = n_{b_i-1}$ come from the fact that the elements $(a_i + 1)$ and $(b_i - 1)$ cannot belong to any other segment unlinked to $(a_i, \ldots, b_i)$.

If for any i those conditions are satisfied $(\underline{n'})$ is a residual segment, by hypothesis.

Now, let us choose a segment which does not contain zero : (a_j, b_j). Since by the Equation (1.17) $n_{a_j+1} = n_{a_j}$ and $n_{b_j} = n_{b_j-1} - 1$, adding only $(a_j, \ldots, b_j)$ yields equations as (1.17) and therefore a residual segment.

If this segment contains zero and is disjoint from the others, then adding all segments or just this one yields the same results on the numbers of zeroes and ones : $n'_0 = n''_0$, $n'_1 = n''_1$, therefore there is no additional constraint under these circumstances.

Secondly, let us consider the case of a chain of inclusions, that, without loss of generality, we denote $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \ldots \supset \mathcal{S}_t$. Starting from $(\underline{n'})$, observe that adding the t linear residual segments yields the following conditions :

$$n'_{a_i+1} = n_{a_i+1} + i - 1 = n'_{a_i} - 1 = n_{a_i} + i - 1$$

$$n'_{b_i} = n_{b_i} + i = n_{b_i-1} - 1 + i = n'_{b_i-1}$$

Then, for any i, we clearly observe $n_{a_i+1} = n_{a_i}$; and $n_{b_i} = n_{b_i-1} - 1$. Assume we only add the segment $(a_1, \ldots, b_1)$, then we observe $n''_{a_1+1} = n''_{a_1} - 1$ and $n''_{b_1} = n''_{b_1-1}$, satisfying the conditions for $(\underline{n''})$ to be a residual segment.

Assume $\mathcal{S}_t$ contains zero, then any $\mathcal{S}_i$ also. Assume there is an obstruction at zero to form a residual segment when adding $t-1$ segments. If adding only $t-1$ zeroes does not form a residual segment, but t zeroes do, we had $n'_0 = \frac{n_1}{2}$. Then $n_0 + t = \frac{n_1}{2} + t = \frac{n_1+2t}{2}$ (the option $n'_1 = n_1 + 2t + 1$ is immediately excluded since there is at most two '1' per segment $\mathcal{S}_i$).

We need to add $2t$ times '1'. Then we need at least $2t-1$ times '2' and $2t-2$ times '3'..etc. Since, $n'_1 = n_1 + 2t$ all $\mathcal{S}_i$'s will contain (10 -1). There is no obstruction at zero while adding solely $\mathcal{S}_1$ (i.e $n_0 + 1 = \frac{n_1+2}{2}$) and since $\mathcal{S}_1 \supset \mathcal{S}_2 \ldots \supset \mathcal{S}_t$ and $\mathcal{S}_1$ needs to contain $a_1 \geq \ell + m$, $\mathcal{S}_1$ can merge with $(\underline{n})$ to form a residual segment.

Finally, it would be possible to observe the case of a residual segment $\mathcal{S}_1$ containing $\mathcal{S}_2$ and $\mathcal{S}_3$ with $\mathcal{S}_2$ and $\mathcal{S}_3$ disjoint (or two-or more- disjoint chains of inclusions). Again, we have :

$$n'_{a_1+1} = n_{a_1+1} = n'_{a_1} - 1 = n_{a_1} + 1 - 1$$

Assume we only add the segment $(a_1, \ldots, b_1)$, then we observe $n''_{a_1+1} = n''_{a_1} - 1$ and $n''_{b_1} = n''_{b_1-1}$, satisfying the conditions for $(\underline{n''})$ to be a residual segment.

□

Remark 9. We show in this remark that if $s_i = \frac{a_i+b_i}{2} = s_j = \frac{a_j+b_j}{2}$, the linear segments $(a_i, \ldots, b_i)$ with $a_i > b_i$ and (a_j, b_j) with $a_j > b_j$ are such that one of them is included in the other (therefore unlinked).

If the length of the segments are the same, they are equal ; without loss of generality let us consider the following case of different lengths :

$$a_i - b_i + 1 > a_j - b_j + 1 \tag{1.18}$$

Since $\frac{a_i+b_i}{2} = \frac{a_j+b_j}{2}$, $a_i + b_i = a_j + b_j$ and from Equation (1.18) $a_i - a_j > b_i - b_j$ replacing b_i by $a_j + b_j - a_i$, and further a_i by $a_j + b_j - b_i$, we obtain :

$$a_i - a_j > a_j + b_j - a_i - b_j \Leftrightarrow a_i > a_j$$
$$a_j + b_j - b_i - a_j > b_i - b_j \Leftrightarrow b_j > b_i$$

Therefore

$$a_i > a_j > b_j > b_i$$

Therefore, the content of the proofs of the next Theorem (60), when considering the case of equal parameters $s_i = s_j$, remain the same.

Theorem 60. *Let π_0 be an irreducible generic representation discrete series of a quasi-split reductive group G. Let us assume σ_ν is in the cuspidal support of a generic discrete series representation τ of a standard Levi subgroup M of G. Let us assume that the cuspidal support of τ satisfies the conditions (CS) (see the Definition 53). Let us take $\underline{s}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$ and denote $\lambda = \nu + \underline{s}$ in $\overline{a_{M_1}^{M}}^{+*}$. Let us assume σ_λ is a residual point for μ^G.*

Then, the unique irreducible generic square-integrable subquotient, π_0, in the standard module $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation.

Proof. Let us assume that Σ_σ^M is a disjoint union of t subsystems of type A and a subsystem of type $\mathcal{T}$.

Let $\underline{s} = (s_1, s_2, \ldots, s_t)$ be ordered such that $s_1 \geq s_2 \geq \ldots \geq s_t \geq 0$ with $s_i = \frac{a_i+b_i}{2}$, for two (half)-integers $a_i \geq b_i$.

Using the depiction of residual points in Subsection 1.5.3, we write the residual point

$$\sigma(\bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n}))$$

where λ reads $\bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n})$.

Let us denote the linear residual segments $(a_i, \ldots, b_i) := \mathcal{S}_i$ and assume that for some indices $i, j \in \{1, \ldots, t\}$, the segments $\mathcal{S}_i, \mathcal{S}_j$ are linked.

By Lemma 40, there exists an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t; \underline{n})$ to $I_{P_1}^G(\sigma((\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t; \underline{n})$. Therefore, if we prove the unique irreducible discrete series subquotient appears as subrepresentation in $I_{P_1}^G(\sigma((\mathcal{S}'_1, \mathcal{S}'_2, \ldots, \mathcal{S}'_t; \underline{n})))$, it will consequently appears as subrepresentation in $I_{P_1}^G(\sigma((\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t; \underline{n})))$. This means we are reduced to the case of the cuspidal support σ_λ being constituted of t *unlinked* segments.

Further, notice that by the above remark [9] when $s_i = s_j$, the segments $\mathcal{S}_i$, and $\mathcal{S}_j$ are *unlinked*. This allows us to treat the case $s_1 = s_2 = \ldots = s_t > 0$ and

$s_1 > s_2 = \ldots = s_t = 0$.

So let us assume all linear segments $(a_i, \ldots, b_i)$ are unlinked.

We prove the theorem by induction on the number t of linear residual segments.

First, $t = 0$, let $P_0 = G$, and π be the generic irreducible square integrable representation corresponding to the dominant residual point $\sigma_\lambda := \sigma(\underline{n}_{\pi_0})$.

$$I_{P_0}^G(\pi) \hookrightarrow I_{P_1}^G(\sigma((\underline{n}_{\pi_0}))$$

By Lemma 34, λ being in the closure of the positive Weyl chamber, the unique irreducible generic discrete series subquotient is necessarily a subrepresentation.

The proof of the step from $t = 0$ to $t = 1$ is Proposition 57.

Assume the result true for any standard module $I_{P'_{\Theta_{\leq t}}}^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{i=1}^t (a_i, \ldots, b_i)(\underline{n})))$ with t or less than t linear residual segments, where $P'_{\Theta_{\leq t}}$ is any standard parabolic subgroup whose Levi subgroup is obtained by removing t or less than t simple (non-extremal) roots from Δ.

We consider now π_0 the unique irreducible generic discrete series subquotient in

$$I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}}) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{i=1}^t (a_i, \ldots, b_i)(a_{t+1}, \ldots, b_{t+1})(\underline{n'})))$$

To distinguish with the case of a discrete series τ of P_{Θ_t}, we denote τ' the irreducible generic discrete series and s' in $a_{M_{\Theta_{t+1}}}*^+$.

Using Lemma 59, we know there is at least one linear segment with index $j \in [1, t+1]$ such that $(a_j, \ldots, b_j)$ can be inserted in $(\underline{n'})$ to form a residual segment. Without loss of generality, let us choose this index to be $t+1$ (else we use bijective intertwining operators on the unlinked segments to set $(a_j, \ldots, b_j)$ in the last position).

Then, there exists a Weyl group element w such that $w((a_{t+1}, \ldots, b_{t+1})(\underline{n'})) = (\underline{n})$ for a residual segment $(\underline{n})$.

Let $M_1 = M_\Theta$ with $\Theta = \bigcup_{i=1}^s \Theta_i$ for some $s > t$ and $M' = M_{\Theta'}$ where $\Theta' = \bigcup_{i=1}^{s-2} \Theta_i \cup \Theta_t \cup \{\underline{\alpha}_t\} \cup \Theta_{t+1}$, if we assume (by convention) that the root $\underline{\alpha}_t$ connects the two connected components Θ_t and Θ_{t+1}.

Since $M' \cap P$ is a maximal parabolic subgroup in M', we can apply the result of Proposition 57 to π' the unique irreducible discrete series subquotient in $I_{P_1 \cap M'}^{M'}(\sigma(a_{t+1}, b_{t+1})(\underline{n'}))$.

Notice that $\Sigma^{M'}$ is a reducible root system, and therefore so is $\Sigma_\sigma^{M'}$; it is because we choose an irreducible component of $\Sigma^{M'}$ that we can apply the result of Proposition 57.

It appears as a subrepresentation in $I_{P_1 \cap M'}^{M'}(\sigma(\underline{n}))$.

Then, since the parameter $\bigoplus_{i=1}^t (a_i, \ldots, b_i)$ corresponds to a central character χ for M', we have :

$$I_{P'}^G(\pi'_\chi) \hookrightarrow I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(\underline{n}))_{\bigoplus_{i=1}^t(a_i,\dots,b_i)}) \cong I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(\underline{n})))$$

By Proposition 57, the subquotient π' appears as a subrepresentation in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},\dots,b_{t+1})(\underline{n'}))$ and therefore in the standard module embedded in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},\dots,b_{t+1})(\underline{n'}))$ by multiplicity one of the irreducible generic piece.

Since the parameter $\bigoplus_{i=1}^t(a_i,\dots,b_i)$ correspond to a central character for M', we have :

$$I_{P'}^G(\pi'_\chi) \hookrightarrow I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},b_{t+1})(\underline{n'}))_{\bigoplus_{i=1}^t(a_i,\dots,b_i)}) \cong I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(a_{t+1},\dots,b_{t+1})(\underline{n'})))$$

We have therefore two options :

Either $I_{P'}^G(\pi'_\chi)$ is irreducible and then it is the unique irreducible generic subrepresentation in

$$I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(a_{t+1},\dots,b_{t+1})(\underline{n'})))$$

$$= I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(a_{t+1},\dots,b_{t+1})(\underline{n'})))$$

and by multiplicity one in $I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}})$.

Either it is reducible, but then its unique irreducible generic subquotient is also the unique irreducible generic subquotient in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(\underline{n})))$.

Then, by induction hypothesis, it embeds as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(\underline{n})))$; and by multiplicity one of the generic piece, also in $I_{P'}^G(\pi'_\chi)$.

Hence it embeds in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\dots,b_i)(a_{t+1},\dots,b_{t+1})(\underline{n'})))$, and therefore in $I_{P_{\Theta_{t+1}}}^G(\tau'_{s'})$ concluding this induction argument, and the proof. □

1.9. The Case of Non-Discrete Series Subquotients and Σ_σ is irreducible

We could have $I_P^G(\tau_{s\tilde{\alpha}})$ reducible without having hypothesis 1 in Lemma 45 satisfied, that is without having $s\tilde{\alpha}$ a pole of the μ function for τ ; i.e the converse of the Lemma 45 doesn't necessarily hold.

It is only in this case that a non-tempered or tempered (but not square-integrable) generic subquotient may occur in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

1.9.1. Proof of the Generalized Injectivity Conjecture for Non-Discrete Series Subquotients

Proposition 61. *Let σ_ν be in the cuspidal support of a generic discrete series representation τ of a maximal Levi subgroup M of a quasi-split reductive group G. Let us take $s\tilde{\alpha}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and denote $\lambda = \nu + s\tilde{\alpha}$ in $\overline{a_{M_1}^M}^{+*}$.*

Let us assume that the cuspidal support of τ satisfies the conditions CS *(see the Definition 53).*

Let us assume σ_λ is not a residual point for μ^G, and therefore the unique irreducible generic subquotient in $I_P^G(\tau_{s\tilde{\alpha}})$ is essentially tempered or non-tempered.

Then, this unique irreducible generic subquotient embeds as a submodule in $I_{P_1}^G(\sigma_\lambda)$ and therefore in the standard module $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof. First, notice that if $s = 0$ the induced module $I_P^G(\tau_{s\tilde{\alpha}})$ is unitary hence any irreducible subquotient is a subrepresentation, in the rest of the proof we can therefore assume $s\tilde{\alpha}$ in $(a_M^*)^+$.

Let us denote π_0 the irreducible generic tempered or non-tempered representation which appears as subquotient in a standard module $I_P^G(\tau_{s\tilde{\alpha}})$ induced from a maximal parabolic subgroup P of G.

We are in the context of the Subsection 1.5.3, and therefore we can write $\lambda := (a, \ldots, b)+(\underline{n})$, for some $a > b$, and residual segment $(\underline{n})$. Here, we assume σ_λ is not a residual point. Then $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma(a, b, \underline{n}))$ has a unique irreducible generic subquotient which is tempered or non-tempered. By Langlands' classification, Theorem 13, and the Standard module conjecture, it has the form $J_{P'}^G(\tau'_{\nu'}) \cong I_{P'}^G(\tau'_{\nu'})$. By Theorem 35, ν' corresponds to the minimal Langlands parameter (this notion was introduced in the Theorem 13) for a given cuspidal support, $\nu' < s\tilde{\alpha}$.

For an explicit description of the parameter ν, given the cuspidal string $(a, b, \underline{n})$, the reader is encouraged to read the analysis conducted in Section G.

The representation τ' (for e.g $St_q|.|^{\nu'} \otimes \pi'$ in the context of classical groups, for a given integer q) corresponds to a cuspidal string $(a', b', \underline{n'})$, and cuspidal representation σ', that is :

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P'_1}^G(\sigma'(a', b', \underline{n'}))$$

By Theorem 2.9 in Bernstein et Zelevinsky 1977, we know the cuspidal data $(P_1, \sigma, (a, b, \underline{n}))$ and $(P'_1, \sigma', \lambda' := (a', b', \underline{n'}))$ are conjugated by an element $w \in W^G$.

By Corollary 47 and since P_1 and P'_1 are standard parabolic subgroups (see Remark 8, we have $P_1 = P'_1$, $w \in W(M_1)$. Any element in $W(M_1)$ decomposes in elementary symmetries with elements in W_σ and $s_{\beta_d} W_\sigma$:

$$\sigma' = w\sigma = \begin{cases} \sigma \text{ if } w \in W_\sigma \\ \text{Else } s_{\beta_d}\sigma \end{cases}$$

Let us assume we are in the context where $\sigma' = s_{\beta_d}\sigma \ncong \sigma$. As explained in the first part of Section 1.8.1, this happens if Σ_σ is of type D.

Let us apply the bijective operator (see Lemma 52) from $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(s_{\beta_d}\sigma)_{\lambda'})$ to $I_{\overline{P_1}\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ and then the bijective map (the definition of the map $t(g)$ has been given in the proof of 16) $t(s_{\beta_d})$ to $I_{s_{\beta_d}(\overline{P_1}\cap(M_1)_{\beta_d})}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'}) = I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'})$.

As explained in Remark 7, $s_{\beta_d}\lambda' = \lambda'$ since λ' is a residual point of type D.

Therefore, we have a bijective map from $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(s_{\beta_d}\sigma)_{\lambda'})$ to $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{\lambda'})$.

The induction of this bijective map gives a bijective map from $I_{P_1'}^{G}(\sigma'(a', b', \underline{n'}))$ to $I_{P_1'}^{G}(\sigma(a', b', \underline{n'}))$.

Now, it is enough to understand how one passes from the cuspidal string $(a', b', \underline{n'})$ to $\sigma(a, b, \underline{n})$ to understand the strategy for embedding the unique irreducible generic subquotient as a subrepresentation in $I_P^G(\tau_{s\tilde{\alpha}})$.

Starting from $(a, b, \underline{n})$, to minimize the Langlands parameter ν', we usually remove elements at the end of the first segment (i.e. the segment $(a, \ldots, b)$) to insert them on the second residual segment, or we enlarge the first segment on the right. This means either $a' < a$, or $b' < b$, or both.

If $a' = a$, and $b' < b$, in particular if $b' < 0$, we have a non-generic kernel operator between $I_{P_1}^{G}(\sigma(a', b', \underline{n'}))$ and $I_{P_1}^{G}(\sigma(a, b, \underline{n}))$ as proved in Lemma 39.

Otherwise, one observes that passing from $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ require certain elements γ, with $a \geq \gamma > a'$, to move up, i.e. from right to left. This means using rank one operators which change $(\gamma + n, \gamma)$ to $(\gamma, \gamma + n)$ for integers $n \geq 1$, those rank one operators may clearly have generic kernel.

In this context, we will rather use the results of Proposition 57.

Consider again $I_{P'}^{G}(\tau'_{\nu'})$ embedded in $I_{P_1}^{G}(\sigma(a', b', \underline{n'}))$. Let us denote π' the unique irreducible generic discrete series subquotient corresponding to the dominant residual point $\sigma((\underline{n'}))$:

Let $M'' = M_{\Delta - \{\underline{\alpha_1}, \ldots, \underline{\alpha_{a-b+1}}\}}$ be a standard Levi subgroup, we have :

$$\pi' \hookrightarrow I_{P_1\cap M''}^{M''}((\sigma((\underline{n'})))$$

Since the character corresponding to the linear residual segment $(a', \ldots, b')$ is central for M'', we write :

$$\pi'_{(a', \ldots, b')} \hookrightarrow I_{P_1\cap M''}^{M''}(\sigma((a', \ldots, b') + (\underline{n'}))) \cong I_{P_1\cap M''}^{M''}(\sigma(\underline{n'}))_{(a', \ldots, b')}$$

Since $\tau'_{\nu'}$ is irreducible (and generic), we also have $\tau'_{\nu'} \hookrightarrow I_{P_1\cap M'}^{M'}(\sigma((a', \ldots, b') + (\underline{n'}))$ we know :

$$\tau'_{\nu'} \hookrightarrow I_{P''}^{M'}(\pi'_{(a', \ldots, b')}) \hookrightarrow I_{P_1\cap M'}^{M'}(\sigma((a', \ldots, b') + (\underline{n'}))) \tag{1.19}$$

By the generalized injectivity conjecture for square-integrable subquotient (Proposition 57), any standard module embedded in $I_{P_1\cap M''}^{M''}(\sigma((\underline{n'})))$ has π' as subrepresentation. We may therefore embed π' as subrepresentation in

$$I_{P_1\cap M''}^{M''}((w_\flat\sigma)((a^\flat, b^\flat, \underline{n}^\flat)))$$

with $w_\flat\sigma \cong \sigma$, and therefore inducing Equation 1.19 to G

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P_1}^G((w_\flat\sigma)((a', \dots, b') + (a^\flat, b^\flat)(\underline{n}^\flat))$$

The sequence $(a^\flat, b^\flat, \underline{n}^\flat)$ is chosen appropriately to have an intertwinning operator with non-generic kernel from $I_{P_1}^G(\sigma((a', \dots, b') + (a^\flat, b^\flat, \underline{n}^\flat))$ to $I_{P_1}^G(\sigma(a, b, \underline{n}))$.

The unique irreducible generic subrepresentation $I_{P'}^G(\tau'_{\nu'})$ in $I_{P_1}^G(\sigma(a, b, \underline{n}))$ cannot appear in the kernel and therefore appears in the image of this operator. It therefore appears as a subrepresentation in $I_{P_1}^G(\sigma(a, b, \underline{n}))$ and by multiplicity one of the generic piece in $I_{P_1}^G(\sigma(a, b, \underline{n}))$, it also appears as subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$. □

Theorem 62. *Let σ_ν be in the cuspidal support of a generic discrete series representation τ of a standard Levi subgroup M of a quasi-split reductive group.*

Let us take $\underline{s}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$ and denote $\lambda = \nu + \underline{s}$ in $\overline{a_{M_1}^M}^{+*}$. Let us assume that σ_λ is not a residual point for μ^G and that the unique irreducible generic subquotient satisfies the conditions CS *(see the Definition 53).*

Then, the unique irreducible generic in $I_P^G(\tau_{\underline{s}})$ (which is essentially tempered or non-tempered) embeds as a subrepresentation in $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof. First, notice that, by the Remark 9, when $s_i = s_j$ the segments $\mathcal{S}_i$, and $\mathcal{S}_j$ are *unlinked.*

Using the argument given in Subsection 1.5.3, we write σ_λ as $\sigma(\bigoplus_{i=1}^t (a_i, \dots, b_i)(\underline{n}))$, where λ reads $\bigoplus_{i=1}^t (a_i, \dots, b_i)(\underline{n})$.

The proof goes along the same inductive line than in the proof of Proposition 60.

The case of $t = 1$ is Proposition 61. That is, given a cuspidal support (P_1, σ_λ), for any standard module induced from a maximal parabolic subgroup $P : I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$, the unique irreducible generic subquotient is a subrepresentation. We use an induction argument on the number t of linear residual segments obtained when removing t simple roots to define the Levi subgroup $M \subset P$. Considering that an essentially tempered or non-tempered irreducible generic subquotient in a standard module with t linear residual segments $I_{P_{\Theta_t}}^G(\tau_{\underline{s}})$ is necessarily a subrepresentation ; one uses the same arguments than in the proof of Theorem 60 to conclude that a tempered or non-tempered irreducible generic subquotient in a standard module with $t+1$ linear residual segments $I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}})$ is a subrepresentation, therefore proving the theorem. □

Eventually, we now consider the generic subquotients of $I_P^G(\gamma_{s\tilde{\alpha}})$ when γ is a generic irreducible tempered representation.

Corollary 63 (Standard modules)**.** *Let G be a quasi-split reductive group of type A, B, C or D and let us assume Σ_σ is irreducible.*

The unique irreducible generic subquotient of $I_P^G(\gamma_{\underline{s}})$ when γ is a generic irreducible tempered representation of a standard Levi M is a subrepresentation.

Proof. Let $P = MU$.

By Theorem 50, as a tempered representation of M, γ appears as a subrepresentation of $I_{P_3\cap M}^M(\tau)$ for some discrete series τ and standard parabolic $P_3 = M_3U$ of G; τ is generic irreducible representation of the Levi subgroup M_3, therefore

$$I_P^G(\gamma_{\underline{s}}) \hookrightarrow I_P^G(I_{M\cap P_3}^M(\tau))_{\underline{s}} \cong I_{P_3}^G(\tau_{\underline{s}})$$

where P_3 is not necessarily a maximal parabolic subgroup of G. Since $\underline{s}$ is in $(a_M^*)^+$, $\underline{s}$ is in $\overline{(a_{M_3}^*)^+}$. Let us write this parameter $\overline{\underline{s}}$ when it is in $\overline{(a_{M_3}^*)^+}$.

The unique irreducible generic subquotients of $I_P^G(\gamma_{\underline{s}})$ are the unique irreducible generic subquotients of $I_{P_3}^G(\tau_{\overline{\underline{s}}})$, where $\overline{\underline{s}}$ is in $\overline{(a_{M_3}^*)^+}$. Since P_3 is not a maximal parabolic subgroup of G, we may now use Theorems 60 and 62 with $\overline{\underline{s}}$ in $\overline{(a_{M_3}^*)^+}$ to conclude that these unique irreducible generic subquotients, whether square-integrable or not, are subrepresentations. □

1.10. The case Σ_σ reducible

Let us recall that the set Σ_σ is a root system in a subspace of $a_{M_1}^*$ (cf. Silberger 1981 3.5) and we assume that the irreducible components of Σ_σ are all of type A, B, C or D. In Proposition 24, we have denoted for each irreducible component $\Sigma_{\sigma,i}$ of Σ_σ, by $a_{M_1}^{M_i*}$ the subspace of $a_{M_1}^{G*}$ generated by $\Sigma_{\sigma,i}$, by d_i its dimension and by $e_{i,1}, \ldots, e_{i,d_i}$ a basis of $a_{M_1}^{M_i*}$ (resp. of a vector space of dimension $d_i + 1$ containing $a_{M_1}^{M_i*}$ if $\Sigma_{\sigma,i}$ is of type A) so that the elements of the root system $\Sigma_{\sigma,i}$ are written in this basis as in Bourbaki, *Groupes et Algèbres de Lie, Chapitre 4,5, et 6*.

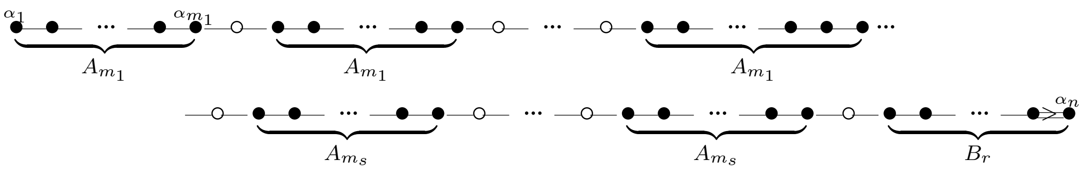
The following result is analogous to Proposition 1.10 in Heiermann 2011.

Proposition 64. *Let $P_1' = M_1U_1'$, and $P_1 = M_1U_1$. If the intersection of $\Sigma(P_1) \cap \Sigma(\overline{P_1'})$ with Σ_σ is empty, the operator $J_{P_1'|P_1}$ is well defined and bijective on $\mathcal{O}$.*

Proof. The operator $J_{P_1'|P_1}$ is decomposed in elementary operators which come from intertwining operators relative to $(M_1)_\alpha$ with $\alpha \notin \Sigma_\sigma$, so it is enough to consider the case where P_1 is a maximal parabolic subgroup of G and $P_1' = \overline{P_1}$. Then, if $\alpha \notin \Sigma_\sigma$ and by the same reasoning than in the previous Lemma 52, the operator $J_{P_1'|P_1}$ is well defined and bijective at any point on $\mathcal{O}$. □

Let G be a quasi-split reductive group over F, π_0 is an irreducible generic representation whose cuspidal support contains the representation σ_λ of a standard Levi subgroup M_1, $\lambda \in a_{M_1}^*$ and σ an irreducible unitary cuspidal generic representation.

In this subsection, we consider the case of a *reducible* root system Σ_σ. As explained in Appendix E, this case occurs in particular when Σ_Θ (see the notations in Appendix E) is reducible, and then Θ has connected components of type A of different lengths. An example is the following Dynkin diagram for Θ :



Let us assume Θ is a disjoint union of components of type A_{m_i} $i = 1 \dots s$, where each component of type A_{m_i} appears d_i times. Set $m_i = k_i - 1$.

Let us denote $\Delta^i_{M_1} = \{\alpha_{i,1}, \dots, \alpha_{i,d_i}\}$ the non-trivial restrictions of roots in Σ, generating the set $a_{M_1}^{M^i *}$. Similar to the case of Σ_σ irreducible, we may have $\Delta_{\sigma_i} = \{\alpha_{i,1}, \dots, \beta_{i,d_i}\}$ where β_{i,d_i} can be different from α_{i,d_i} in the case of type B, C or D. For any $i \neq s$, the pre-image of the root α_{i,d_i} is *not* simple.

Indeed, for instance, in the above Dynkin diagram, the first root 'removed' is $e_{k_1} - e_{k_1+1}$, the second is $e_{2k_1} - e_{2k_1+1}$;...etc; they are simple roots and their restrictions to A_{M_1} are roots of $\Delta^1_{M_1}$ (the generating set of $a_{M_1}^{M^1 *}$); the last root to consider is $e_{k_1 d_1} - e_{n-r+1}$ which restricts to $e_{k_1 d_1}$; then the preimage of $e_{k_1 d_1}$ is not simple.

However, since $e_{n-r} - e_{n-r+1}$ restricts to e_{n-r}; the pre-image of α_{s,d_s} is simple.

The Levi subgroup M^i is defined such that $\Delta^{M^i} = \Delta^{M_1} \cup \left\{\underline{\alpha_{i,1}}, \dots, \underline{\alpha_{i,d_i}}\right\}$ where $\Delta^i_{M_1} = \{\alpha_{i,1}, \dots, \alpha_{i,d_i}\}$.

It is a *standard* Levi subgroup for $i = s$.

Furthermore, since $\Sigma_{\sigma,i}$ generates $a_{M_1}^{M^i *}$ and is of rank d_i, the semi-simple rank of M^i is $d_i + rk_{ss}(M_1)$. Since $\Sigma_{\sigma,i}$ is irreducible, an equivalent of Proposition 46 is satisfied for M^i.

Proposition 65. *Let π_0 be an irreducible generic representation of a quasi-split reductive group G, and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where M is a maximal Levi subgroup (and α is not an extremal simple root on the Dynkin diagram of Σ) of G and τ is an irreducible generic discrete series of M. Let us assume Σ_σ is reducible.*

Then π_0 is a subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$.

Proof. Let us repeat the initial context :

The representation τ is an irreducible generic discrete series of a maximal Levi subgroup $M = M_\Theta$ such that $I_P^G(\tau_{s\tilde{\alpha}})$ is a standard module. By Heiermann-Opdam's result, $\tau \hookrightarrow I^M_{P_1 \cap M}(\sigma_\nu)$, for $\nu \in (a_{M_1}^{M*})^+$. Then, ν is a residual point for μ^M.

Let us write $\Sigma_\sigma^M = \bigcup_{i=1}^r \Sigma_{\sigma,i}^M$, then the residual point condition is $\dim((a_{M_1}^M)^*) = rk(\Sigma_\sigma^M) = \sum_{i=1}^r d_i^M$, where d_i^M is the dimension of $(a_{M_1}^{M^i})^*$ generated by $\Sigma_{\sigma,i}^M$. The residual point ν decomposes in r disjoint residual segments : $\nu = (\nu_1, \dots, \nu_r) := (\underline{n_1}, \underline{n_2}, \dots, \underline{n_r})$.

Since Σ^M decomposes into two disjoint irreducible components, one of them being of type A, the restrictions of simple roots of this irreducible component of type A in Δ^M generates an irreducible component of Σ_σ of type A, let us denote this A component $\Sigma^M_{\sigma,i}$, $d_i = \mathfrak{b} - \gamma$, and denote $\nu_i + s\tilde{\alpha} := (\mathfrak{b}, \dots, \gamma)$ the twisted residual segment of type A.

Let us further assume that there is one index j such that there exists a residual segment $(\underline{n'_j})$ of length $\mathfrak{b} - \gamma + 1 + d_j$ and type $\mathcal{T}$ (B, C or D) in the Weyl group orbit of $(\mathfrak{b}, \gamma)(\underline{n_j})$ where the residual segment $(\underline{n_j})$ is of the same type as $\mathcal{T}$.

Since all intertwining operators corresponding to rank one operators associated to s_β for $\beta \notin \Delta_\sigma$ are bijective (see Lemma 52), all intertwining operators interchanging any two residual segments $(\underline{n_k})$ and $(\underline{n_{k'}})$ are bijective. Therefore, we can interchange the positions of all residual segments (or said differently interchange the order of the irreducible components for $i = 1, \dots, r$) and therefore set $(\mathfrak{b}, \dots, \gamma)(\underline{n_j})$ in the last position, i.e we set $i = r-1, j = r$.

When adding the root α to Θ (when inducing from M to G), we form from the disjoint union $\Sigma^M_{\sigma,r-1} \bigcup \Sigma^M_{\sigma,r}$ the irreducible root system that we denote $\Sigma_{\sigma,r}$.

The Levi subgroup M^r is the smallest standard Levi subgroup of G containing M_1, the simple root α and the set of simple roots whose restrictions to A_{M_1} lie in $\Delta^r_{M_1}$. It is a group of semi-simple rank $d_r + rk_{ss}(M_1)$.

We may therefore apply the results of the previous subsections with Σ_σ irreducible to this context :

Let us assume first the unique irreducible generic subquotient π is discrete series.

From the result of Heiermann-Opdam, we have :

$$\pi \hookrightarrow I^{M^r}_{P_1 \cap M^r}(\sigma(\underline{n'_r}))$$

where the residual segment $(\underline{n'_r})$ is the dominant residual segment in the W_σ orbit of $(\mathfrak{b}, \gamma, \underline{n_r})$.

The unramified character χ corresponding to the remaining residual segments (n_k)'s, $k \neq r-1, r$ is a central character of M^r. Then :

$$\pi_\chi \hookrightarrow I^{M^r}_{P_1 \cap M^r}(\sigma(\underline{n'_r}))_{\bigoplus_{j \neq r-1, r}(\underline{n_j})}$$

As a result :

$$\pi_0 \hookrightarrow I^G_{P_R}(\pi_\chi) \hookrightarrow I^G_{P_1}(\sigma(\bigoplus_{j \neq r}(\underline{n_j}) + (\underline{n'_r}))) \tag{1.20}$$

In Equation 1.20, we claim π_0 embeds first in $I^G_{P_1}(\sigma(\bigoplus_{j \neq r} \underline{n_j}) + (\underline{n'_r})))$ by the Heiermann-Opdam embedding result (since the residual segment $\bigoplus_{j \neq r}(\underline{n_j}) + (\underline{n'_r})$ corresponds to a character in $\overline{(a^*_{M_1})^+}$), therefore it should embed in $I^G_{P_R}(\pi_\chi)$ by multiplicity one of the irreducible generic piece.

Applying our conclusion in the case of irreducible root system (in Proposition 57) to $\Sigma_{\sigma,r}$, we embed π in the induced module $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))$ as a subrepresentation (and therefore in a standard module $I_{P\cap M^r}^{M^r}(\tau_{\frac{\mathscr{b}+\gamma}{2}})$ embedded in $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r})))$.

$$\pi_\chi \hookrightarrow I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))_{\bigoplus_{j\neq r-1,r}(\underline{n_j})} \cong I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r})+\bigoplus_{j\neq r-1,r}(\underline{n_j}))$$

Therefore :

$$\pi_0 \hookrightarrow I_{P_R}^G(\pi_\chi) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{j\neq r}(\underline{n_j})+(\mathscr{b},\gamma,\underline{n_r}))$$

In case π is tempered or non-tempered, and embeds (as a subrepresentation) in $I_{P_1\cap M^r}^{M^r}((\sigma(\mathscr{b}',\gamma',\underline{n_r'}))$, we had shown in Proposition 61 there existed an intertwinning operator with non-generic kernel sending π in $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))$.

Since the other remaining residual segments (n_k')'s, $k\neq r-1,r$ do not contribute when minimizing the Langlands parameter ν', the unique irreducible generic subquotient in

$$I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k})+(\mathscr{b},\gamma,\underline{n_r})))$$

embeds in

$$I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k})+(\mathscr{b}',\gamma',\underline{n_r'})))$$

and we can use the inducting of the previously defined intertwining operator to send this generic subquotient as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k})+(\mathscr{b},\gamma,\underline{n_r})))$. We conclude the argument as usual : by multiplicity one, the generic piece also embeds as a subrepresentation in the standard module. □

Proposition 66. *Let π_0 be an irreducible generic representation and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\tau_s)$, where M is obtained by removing t simple roots from the Dynkin diagram of G, $s=(s_1,\ldots,s_t)$ such that $s_1\geq s_2\geq\ldots\geq s_t$ and τ is an irreducible generic discrete series.*

Then it is a subrepresentation.

Proof. The representation τ is an irreducible generic discrete series of a non-maximal Levi subgroup M such that $I_P^G(\tau_s)$ is a standard module. By Heiermann-Opdam's result, $\tau\hookrightarrow I_{P_1\cap M}^M(\sigma_\nu)$, for $\nu\in\overline{(a_{M_1}^{M\,*})^+}$. Then, ν is a residual point for μ^M.

Let us denote $M=M_\Theta$. Then $\Theta=\bigcup_{i=1}^{t+1}\Theta_i$ where Θ_i, for $i\in\{1,\ldots,t\}$ is of type A.

Since M_1 is a standard Levi subgroup of G contained in M, we can write $\Sigma_\sigma^M=\bigcup_{i=1}^{t+r}\Sigma_{\sigma,i}^M$, then the residual point condition is $\dim((a_{M_1}^M)^*)=rk(\Sigma_\sigma^M)=\sum_{i=1}^{r+t}d_i^M$, where d_i^M is the dimension of $(a_{M_1}^{M^i})^*$ generated by $\Sigma_{\sigma,i}^M$. The residual point ν decomposes in t linear residual segments along with r residual segments : $\nu=(\nu_1,\ldots,\nu_{r+t}):=(\underline{n_1},\underline{n_2},\ldots,\underline{n_{r+t}})$.

Adding the twist $s = (s_1, \ldots, s_t)$, we obtain a parameter λ in $(a_{M_1}^G)^*$ composed of t twisted linear residual segments $\{(a_i, \ldots, b_i)\}_{i=1}^t$ and r residual segments $(\underline{n_1}, \underline{n_2}, \ldots, \underline{n_r})$.

Let us first assume that λ is a residual point.

This means all linear residual segments can be incorporated in the r residual segments of type $\mathcal{T}$ to form residual segments $\left\{(\underline{n'_j})\right\}_{j=1}^r$ of type $\mathcal{T}$ and length d_i such that $\sum_i d_i = d$ where d is $rk_{ss}(G) - rk_{ss}(M_1) = \dim a_{M_1} - \dim a_G$. It is also possible that, as twisted linear residual segments they are already in a form as in Proposition 58. In that case, the linear residual segment need not be incorporated in any residual segment of type $\mathcal{T}$.

Furthermore, as in the proof of Theorem 60, we can reduce our study to the case of unlinked residual linear segments.

By Heiermann-Opdam's Proposition (12) :

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_j \underline{n'_j}))$$

Let us consider the last irreducible component $\Sigma_{\sigma,r}$ of Σ_σ and the residual segment $(\underline{n'_r})$ associated to it.

Let us assume this irreducible subsystem is obtained from some subsystems $\Sigma_{\sigma,i}^M$ of type A denoted $A_q, \ldots, A_s$ and one of type $\mathcal{T}$ when inducing from M to G

$$\{A_q, \ldots, A_s\} \leftrightarrow \{\mathcal{T}\} \tag{1.21}$$

$$\{(b_{r,q}, \ldots, \gamma_{r,q}), \ldots, (b_{r,s}, \ldots, \gamma_{r,s})\} \leftrightarrow \left\{(\underline{n_r})\right\} \tag{1.22}$$

The Levi subgroup M^r is the smallest standard Levi subgroup of G containing M_1, s simple roots (among the t simple roots in $\Delta - \Theta$) and the set of roots whose restrictions to A_{M_1} lie in $\Delta_{M_1}^r$. It is a group of semi-simple rank $d_r + rk_{ss}(M_1)$.

We may therefore apply the results of the previous subsections with Σ_σ irreducible to this context : the unique irreducible generic discrete series, π, in the induced module $I_{P_1\cap M^r}^{M^r}(\sigma(\bigoplus_{j=q}^s(b_{r,j}, \gamma_{r,j}) + (\underline{n_r}))$ is a subrepresentation.

As in the proof of the previous Proposition 65, since π also embeds in $I_{P_1\cap M^r}^{M^r}(\sigma(\underline{n'_r}))$, when we add the twist by the central character corresponding to $\bigoplus_{k\neq r}(\underline{n'_k})$, we obtain :

$$\pi_0 \hookrightarrow I_P^G(\pi_\chi) \hookrightarrow I_{P_R}^G(I_{P_1\cap M^r}^{M^r}(\sigma(\bigoplus_{j=k}^s(b_{r,j}, \ldots, \gamma_{r,j}) + (\underline{n_r}))_{\bigoplus_{k\neq r-1,r}(\underline{n'_k})}))$$

In case π is non-tempered, and embeds (as a subrepresentation) in $I_{P_1\cap M^r}^{M^r}((\sigma(b', \gamma', \underline{n''_r}))$, we had shown in Proposition 61 there existed an intertwinning operator with non-generic kernel sending π in $I_{P_1\cap M^r}(\sigma(b, \gamma, \underline{n_r}))$.

Since the other remaining residual segments (n'_k)'s, $k \neq r$ do not contribute when minimizing the Langlands parameter ν', the unique irreducible generic subquotient in

$$I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k}) + (\mathscr{b},\gamma,\underline{n_r})))$$

embeds in

$$I_{P_R}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k}) + (\mathscr{b}',\gamma',\underline{n'_r})))$$

and we can use the inducting of the previously defined intertwining operator to send this generic subquotient as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k})+(\mathscr{b},\gamma,\underline{n_r})))$.

Then

$$\pi_0 \hookrightarrow I_{P_R}^G(\pi_\chi) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k}) + \bigoplus_{j=q}^{s}(\mathscr{b}_{r,j},\gamma_{r,j}) + (\underline{n_r})))$$

We conclude the argument as usual : by multiplicity one, the generic piece also embeds as a subrepresentation in the standard module.

Using bijective interwining operators, we now reorganize this cuspidal support so as to put the linear residual segments $\bigoplus_{j=q}^{s}(\mathscr{b}_{r,j},\gamma_{r,j})$ on the left-most part and $\Sigma_{\sigma,r-1}$ in the right-most part. The residual segment $(\underline{n'_{r-1}})$ is (possibly) again formed of some linear residual segments $(\mathscr{b}_i,\gamma_i)$ and the residual segment $(\underline{n_{r-1}})$. We argue just as above. Since the linear residual segments are linked, we can reorganize them so as to insure $s_1 \geq s_2 \geq \ldots s_t$.

Eventually repeating this procedure,

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(\mathscr{b}_i,\gamma_i) + \bigoplus_{j=1}^{r}(\underline{n_j})))$$

Further, by multiplicity one, the generic piece also embeds as a subrepresentation in the standard module.

□

Corollary 67. *Let π_0 be an irreducible generic representation and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\gamma_{\underline{s}})$, where M is a standard Levi subgroup of G.*

Then it is a subrepresentation.

Proof. Let $P = MU$.

We argue as in the Corollary 63 : using the Theorem 50, the tempered representation of M, γ, appears as a subrepresentation of $I_{P_3\cap M}^M(\tau)$ for some discrete series τ and standard parabolic $P_3 = M_3U$ of G ; τ is a generic irreducible representation of the standard Levi subgroup M_3, therefore

$$I_P^G(\gamma_{\underline{s}}) \hookrightarrow I_P^G(I_{M\cap P_3}^M(\tau))_{\underline{s}} \cong I_{P_3}^G(\tau_{\underline{s}})$$

where P_3 is not necessarily a maximal parabolic subgroup of G.

Since $\underline{s}$ is in $(a_M^*)^+$, $\underline{s}$ is in $\overline{(a_{M_3}^*)^+}$. Let us write this parameter $\overline{\underline{s}}$ when it is in $\overline{(a_{M_3}^*)^+}$.

The unique irreducible generic subquotients of $I_P^G(\gamma_{\underline{s}})$ are the unique irreducible generic subquotients of $I_{P_3}^G(\tau_{\overline{\underline{s}}})$, where $\overline{\underline{s}}$ is in $\overline{(a_{M_3}^*)^+}$. Since P_3 is not a maximal parabolic subgroup of G, we use the result of the previous Proposition 66. $\square$

APPENDICES

A. Background

A.1. Structure of parabolic subgroups

For further reading on the materials briefly presented in this Section, see for instance Kim 2004 and Murnaghan 2005.

Let **G** be an arbitrary connected reductive algebraic group over F, let $\mathbf{A}_0$ be a maximal split torus over F and $\Sigma = \Sigma(\mathbf{G}, \mathbf{A}_0)$.

Theorem 68. *There is a one to one correspondence between Borel subgroups containing $\boldsymbol{A}_0$ and fundamental system Δ of Σ. The correspondence is*

$$\boldsymbol{B} = B_\Delta \leftrightarrow \Delta \subset \Sigma$$

$B_\Delta = \boldsymbol{A}_0 \prod_{\alpha \in \Sigma^+} U_\alpha$ where Σ^+ is the set of positive roots in Sigma determined by Δ.

Definition 69. A closed subgroup of **G** which contains a Borel subgroup is called a parabolic subgroup of **G**.

Theorem 70. *There is a one to one correspondence between parabolic subgroups $\boldsymbol{P} = P_\theta$ containing B_Δ and subsets $\theta \subset \Delta$. The correspondence is*

$$\boldsymbol{P} = P_\theta \leftrightarrow \theta \subset \Delta$$

$$P_\theta = G(\Sigma_\theta).A_\theta U_\theta^+ = M_\theta N_\theta$$

where $M_\theta = G(\Sigma_\theta)A_\theta$ is the Levi subgroup of P_θ and $N_\theta = U_\theta^+ = \prod_{\alpha \in \Sigma^+ - \Sigma_\theta^+} U_\alpha$ is the unipotent radical of P_θ.

$\Sigma_\theta^+ = \{\theta\}_{\mathbb{Z}} \cap \Sigma^+$. Here $A_\theta = \bigcap_{\alpha \in \theta} (ker(\alpha))^0$,the subtorus of $\boldsymbol{A}_0$ annihilated by θ and $G(\Sigma_\theta)$ is the subgroup generated by $U_\alpha, \alpha \in \Sigma_\theta = \{\theta\}_{\mathbb{Z}} \cap \Sigma$.

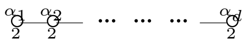
In particular, the Borel subgroup **B** corresponds to the empty set in Δ. Also, note that if $\theta_1 \subset \theta_2 \subset \Delta$, then $P_{\theta_1} \subset P_{\theta_2}$.

The torus A_θ is split over F, and $M_\theta := Z_G(A_\theta)$. This group is reductive and defined over F. Its maximal split torus is again A_0. The set of roots of M_θ is Σ_θ, which is by definition the set of all roots in $\Sigma (= \Sigma_\Delta)$ generated by θ.

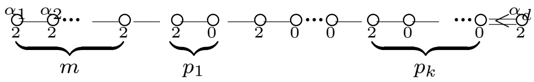
B. Weighted Dynkin diagrams

The diagrams presented here are also presented in Carter's book CARTER 1985, page 175.

A_d

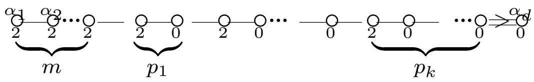


C_d



with $m + p_1 + \dots p_k + 1 = d$, $p_1 = 2, p_{i+1} = p_i$ or $p_i + 1$ for each i. ($k = 0$, $m = l - 1$ is a special case)

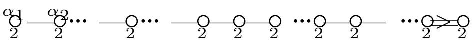
B_d



with $m + p_1 + \dots p_k = d$, $p_1 = 2, p_{i+1} = p_i$ or $p_i + 1$ for $i = 1, 2, \dots, k-2$ and

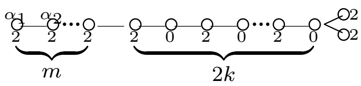
$$p_k = \begin{cases} \frac{p_{k-1}}{2} \text{ if } p_{k-1} \text{ is even} \\ \frac{p_{k-1}-1}{2} \text{ if } p_{k-1} \text{ is odd} \end{cases}$$

In addition the diagram :

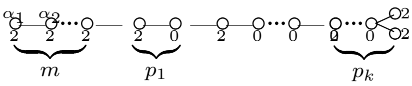


is distinguished.

D_d



with $m + 2k + 2 = d$, and those of the form



with $m + p_1 + \dots p_k = l$, $p_1 = 2, p_{i+1} = p_i$ or $p_i + 1$ for $i = 1, 2, \dots, k-2$ and

$$p_k = \begin{cases} \frac{p_{k-1}}{2} \text{ if } p_{k-1} \text{ is even} \\ \frac{p_{k-1}+1}{2} \text{ if } p_{k-1} \text{ is odd} \end{cases}$$

B.1. Examples of *Set of Jumps* and residual segments

Example 12 (B_9)**.** Let $d_i'= 9$. Then $2d_i'+1$ is 19, and we decompose 19 into distinct odd integers : 19 ; 11+7+1 ; 13+5+1 ; 15+3+1. So they are four different weighted Dynkin diagrams for B_9. The integers a_i's are respectively $\{9\}$; $\{5, 3\}$; $\{6, 2\}$; $\{7, 1\}$.

Example 13 (D_9)**.** Then $2d_i'$ is 18, and we decompose 18 into distinct odd integers : 1 + 17 ; 15+3 ; 11+7 ; To each of these partitions correspond the Weyl group orbit of a residual point and therefore a residual segment. The regular orbit (since the exponents of the associated residual segment form a regular character of the torus) correspond to 1+17. It is simply $(8, 7, \dots 1, 0)$.

The other residual segments are : (765432110) ;(654322110) ; (543322110) ; (4 32 211 100) and the corresponding Jordan blocks are $\{15, 3\}$; $\{13, 5\}$; $\{11, 7\}$; $\{9, 5, 3, 1\}$.

C. The obstruction to the dominance of the parameter $\lambda \in a^*_{M_1}$ with respect to P_1

With the setting of Section 1.3, P is a maximal parabolic subgroup therefore corresponding to a subset of roots θ consisting of all roots of Δ minus α. Also, recall that following Heiermann et Opdam 2009, let us denote $a^{M*}_{M_1} = \mathbb{R}\Sigma^M \subset a^{G*}_{M_1}$, where Σ^M are the roots in Σ which are in M (with basis Δ^M).

We will abusively denote $s\tilde{\alpha}$, a complex multiple of a fundamental weight of α.

Asssuming $P_1 = P_0 = P_\emptyset$, to say λ is dominant for P_1 means $\left\langle \lambda, \check{\beta} \right\rangle \geq 0$ for all β in $\Delta - \emptyset$, and $\left\langle \lambda, \check{\beta} \right\rangle = 0$ for all β in $\emptyset$. This corresponds to $\lambda \in \overline{a^*_{M_1}}^+$.

Also in this context, when considering the positive Weyl chamber relative to M, $\overline{a^{M*}_{M_1}}^+$, one should consider all the roots except α since α is precisely the root « which is not in Lie(M) ».

Let ν be in the closed positive Weyl chamber relative to M : $\nu \in \overline{a^{M*}_{M_1}}^+$.

We have $\left\langle \nu + s\tilde{\alpha}, \check{\beta} \right\rangle = \left\langle \nu, \check{\beta} \right\rangle + s \geq 0$ for all β in $\Delta - \emptyset$ except α.

The root α however leads to : $\langle \nu + s\tilde{\alpha}, \check{\alpha} \rangle = s + \langle \nu, \check{\alpha} \rangle$ where the second element could clearly be negative, since $\nu \in \overline{a^{M*}_{M_1}}^+$.

Given that $s\tilde{\alpha}$ is a residual point for μ^G and τ (and therefore we have shown that $\nu + s\tilde{\alpha}$ is a residual point for σ), we would like to observe when $\nu + s\tilde{\alpha}$ is in $\overline{a^*_{M_1}}^+$, i.e when $\nu + s\tilde{\alpha}$ is a dominant residual point.

Consider the example of GL_4, $M = GL_2 \times GL_2$, and $M_1 = GL_1^4$. The set of simple roots is $\Delta := \{\alpha_1, \alpha_2, \alpha_3\}$, and the maximal parabolic subgroup M is $M_\theta = M_{\Delta - \alpha_2}$.

Take τ to be the product of two Steinberg, then $\sigma = \mathbf{1}$, and $\chi_\nu = |.|^{1/2}|.|^{-1/2} \otimes |.|^{1/2}|.|^{-1/2}$.

For ν to be dominant with respect to $P_1 = P_\theta = P_\emptyset$, we need $\langle \nu, \check{\alpha_i} \rangle \geq 0$ for all α_i in $\Delta - \emptyset$.

But notice $\langle \nu, \check{\alpha_2} \rangle = -1/2 - 1/2 = -1$ and therefore $\nu \notin \overline{a_{M_1}^{*}}^{+}$.

However, $\nu \in \overline{a_{M_1}^{M*}}^{+}$, that is : $\langle \nu, \check{\alpha_i} \rangle \geq 0$ for α_1 and α_3 (roots that are also in M).

Now, we may find some condition on s such that when considering $I_{P_1}^G(\sigma_{\nu + s\tilde{\alpha}})$, $\nu + s\tilde{\alpha}$ will be dominant for P_1. We consider $I_P^G(St_2|det|^{s_1} \otimes St_2|det|^{s_2})$ as standard module and $s\tilde{\alpha}$ implies $s_1 > s_2$.

$I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}}) := I_{P_1}^G(|.|^{s_1+1/2}|.|^{s_1-1/2} \otimes |.|^{s_2+1/2}|.|^{s_2-1/2})$

For $\nu + s\tilde{\alpha}$ to be dominant, we need :

$$s_1 - 1/2 - (s_2 + 1/2) > 0$$

$$s_1 - s_2 > 1$$

This elementary exemple with GL_4 can easily be extended to $G = GL_{n+m}$ with $\sigma = \mathbf{1}, \chi_\nu = |.|^{\frac{n-1}{2}} \ldots |.|^{\frac{1-n}{2}} \otimes |.|^{\frac{m-1}{2}} \ldots |.|^{\frac{1-m}{2}}$.

$St_n \times St_m \hookrightarrow I_{P_1 \cap M}^M(\sigma_\nu), \nu \in \overline{a_{M_1}^{M*}}^{+}$.

Adding the twist by $s\tilde{\alpha}$ gives :

$$I_{P_1}^G(|.|^{s_1+\frac{n-1}{2}} \ldots |.|^{s_1+\frac{1-n}{2}} \otimes |.|^{s_2+\frac{m-1}{2}} \ldots |.|^{s_2+\frac{1-m}{2}})$$

To have the dominance relative to P_1, we need in particular :

$$s_1 + \frac{1-n}{2} - (s_2 + \frac{m-1}{2}) > 0$$

$$s_1 - s_2 > \frac{n+m-2}{2}$$

The unique residual point of $I_{P_1}^G(|.|^{\frac{n+m-1}{2}}|.|^{\frac{m+n-2}{2}} \ldots |.|^{+\frac{1-n-m}{2}})$ gives the associated discrete series St_{m+n} as a subrepresentation.

To have $I_{P_1}^G(|.|^{s_1+\frac{n-1}{2}} \ldots |.|^{s_1+\frac{1-n}{2}} \otimes |.|^{s_2+\frac{m-1}{2}} \ldots |.|^{s_2+\frac{1-m}{2}}) \cong I_{P_1}^G(|.|^{\frac{n+m-1}{2}}|.|^{\frac{m+n-2}{2}} \ldots |.|^{+\frac{1-n-m}{2}})$, we need $s_1 = \frac{m}{2}$ and $s_2 = \frac{-n}{2}$, but then clearly $s_1 - s_2 = \frac{m+n}{2} > \frac{n+m-2}{2}$.

Therefore in the case of the linear group the necessary choice of $s_1 = \frac{m}{2}$ and $s_2 = \frac{-n}{2}$, implies $\nu + s\tilde{\alpha}$ is in $\overline{a_{M_1}^{*}}^{+}$, that is $\nu + s\tilde{\alpha}$ is dominant for $P_1 = P_0$.

However, in the case of classical group, consider the following example :

Example 14. We start from a residual segment on $Sp(14)$ given by the vector $\nu_2 = (3221110)$. On the linear part, we have the vector of $\nu_1 := (43210)$ of GL_5. (ν_1, ν_2) is the residual segment of the maximal Levi subgroup M of the form $GL_5 \times Sp(14)$ of $Sp(24)$. To be slightly more general, we fix the cuspidal representation to be ϱ of M_1 rather than the trivial as fixed above. We have the maximal parabolic subgroup $P = MU$. The standard module embeds in the cuspidal support as follows :

$$I_P^G(\mathrm{St}_5(\varrho)|.|^2 \otimes \pi) \hookrightarrow I_{P_1}^G(|.|^4\varrho|.|^3\varrho|.|^2\varrho|.|^1\varrho|.|^0\varrho|.|^3\varrho|.|^2\varrho\ldots|.|^0\varrho)$$

The parameter $\nu + s\tilde{\alpha} := (432103221110)$ is not in the closed positive Weyl chamber $\overline{(\alpha^*_{M_1})^+}$, indeed observe that $\langle \nu + s\tilde{\alpha}, \check{\alpha}_5 \rangle = -3$.

A Weyl group element, $w \in W_\varrho$ will however send $\nu + s\tilde{\alpha}$ to the vector (433222111100) corresponding to a residual point for $Sp(24)$ which is in the closure of the positive Weyl chamber, i.e *a dominant residual point*. As a residual point, it gives a discrete series subquotient ; and therefore the generic irreducible subquotient is a discrete series.

D. Illustrations with linear and classical groups

For the curious reader, we include in this section our reflexions and illustrations of the reasoning developed previously in the context of linear and classical groups (in the sense of Hanzer HANZER 2010 or Moeglin-Tadic MOEGLIN et TADIC 2002).

With the notions of the Subsection 1.5, we have :

Proposition 71. *Let σ_1, σ_2 be two unitary cuspidal representations of $GL_{n_i}(F)$ in two disjoint inertial classes, $s_1, s_2 \in \mathbb{R}$.*

Then the intertwining operator $J_{P|\bar{P}}(\sigma_1|.|^{s_1} \otimes \sigma_2|.|^{s_2})$ between

$$I_{\bar{P}}^G(\sigma_1|.|^{s_1} \otimes \sigma_2|.|^{s_2}) \qquad \text{and} \qquad I_P^G(\sigma_1|.|^{s_1} \otimes \sigma_2|.|^{s_2})$$

is one-to-one.

Proof. Recall

$$J_{P|\bar{P}}(\sigma_1|.|^{s_1}\otimes\sigma_2|.|^{s_2})J_{\bar{P}|P}(\sigma_1|.|^{s_1}\otimes\sigma_2|.|^{s_2}) = \mu^{-1}_{\bar{P}|P}(\sigma_1|.|^{s_1}\otimes\sigma_2|.|^{s_2}) = (\mu^{M_\alpha})^{-1}(\sigma_1|.|^{s_1}\otimes\sigma_2|.|^{s_2})$$

As σ_1 and σ_2 are in two disjoint inertial classes, it is clear that there does not exist any non-trivial element s_α in $W^{M_\alpha}(M)$ such that $s_\alpha(\sigma_1 \otimes \sigma_2) \cong \sigma_1 \otimes \sigma_2$.

Then use Harish-Chandra's Theorem [Theorem 8], point (a), to say that $\mu(\sigma_1 \otimes \sigma_2) \neq 0$. Further, this is also true for $\sigma_1|.|^{s_1}$ and $\sigma_2|.|^{s_2}$ for all $s_1, s_2 \in \mathbb{R}$. Considering the inertial orbit $\mathcal{O}$ of $\sigma_1\otimes\sigma_2$, we therefore have $\mu^{M_\alpha}(\sigma_1\otimes\sigma_2) \neq 0$ on $\mathcal{O}$, i.e $\alpha \notin \Sigma_{\mathcal{O},\mu}$.

Since $\alpha \notin \Sigma_{\mathcal{O},\mu}$, μ^{M_α} is constant by a Proposition of Silberger, recalled as Proposition 1.6 in HEIERMANN 2011. Using Silberger[5.4.2.1] the poles of $J_{P|\bar{P}}$ are representations which become unitary after twisting with an unramified character of G.

The function $\mu J_{P|\bar{P}}$ is regular on any unitary representation of $\mathcal{O}$[WALDSPURGER 2003], $J_{P|\bar{P}}$ must also be regular on $\mathcal{O}$, since it is polynomial on $X_{ur}(G)$.

It is therefore one-to-one on $\mathcal{O}$ since μ is never zero. □

D.1. Generalized Injectivity for GL_n

Theorem 72. *Let $I_P^{GL_N}(St_{n_1}(\rho)|.|^{s_1} \otimes St_{n_2}(\rho)|.|^{s_2})$ be a generic standard module, i.e $s_1 > s_2$, with ρ an irreducible cuspidal representation of GL_n and $N = (n_1 + n_2).n$. Denote $I_{P'}^{GL_N}(\tau'_{\nu'})$ its unique irreducible generic subquotient. Then the unique irreducible generic subquotient appears as a subrepresentation.*

Proof. The cuspidal support of $St_{n_1}(\rho)|.|^{s_1} \otimes St_{n_2}(\rho)|.|^{s_2}$ is characterized by the choice of an irreducible cuspidal representation ρ and two linear residual segments $\mathcal{S}_1 = (a_1, \ldots, b_1)$ and $\mathcal{S}_2 = (a_2, \ldots, b_2)$.

$$I_P^{GL_N}(St_{n_1}(\rho)|.|^{s_1} \otimes St_{n_2}(\rho)|.|^{s_2}) \hookrightarrow I_{P_1}^{GL_N}(\rho_\lambda)$$

where $\lambda := (a_1, \ldots, b_1, a_2, \ldots, b_2)$; $s_1 = \frac{a_1+b_1}{2} > s_2 = \frac{a_2+b_2}{2}$.

There are two options : Either λ is a residual point and the unique irreducible generic subquotient $I_{P'}^{GL_N}(\tau'_{\nu'})$ is discrete series. This generic subquotient embeds in $I_{P_1}^{GL_N}(\rho(a_1, \ldots b_1, a_2, b_2))$ where the dominant residual point $(a_1, \ldots b_1, a_2, b_2)$ shall be a decreasing sequence of integers. This means $b_1 = a_2 + 1$, and therefore λ is the dominant residual point. The irreducible generic discrete series necessarily embeds as a subrepresentation in $I_{P_1}^{GL_N}(\rho_\lambda)$. Else, λ is not a residual point and the unique irreducible generic subquotient $I_{P'}^{GL_N}(\tau'_{\nu'})$ is non-discrete series. In this case, the two segments $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked. By Theorem 35 and Propositions 37 and 38, it corresponds to the irreducible subquotient obtained by taking intersection and union of the two segments.

Denote $\mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{S}_1 \cup \mathcal{S}_2$, intersection and union. Since it is irreducible the induced module $I_{P'}^{GL_N}(Z(\mathcal{S}_1 \cap \mathcal{S}_2) \otimes Z(\mathcal{S}_1 \cup \mathcal{S}_2)) \cong I_{P'}^{GL_N}(Z(\mathcal{S}_1 \cup \mathcal{S}_2) \otimes Z(\mathcal{S}_1 \cap \mathcal{S}_2))$. To conclude we need to study the intertwining operators from $I_{P_1}^{GL_N}(\rho(a_2, b_1, a_1, b_2))$ to $I_{P_1}^{GL_N}(\rho(a_1, b_1, a_2, b_2))$. Since $a_1 \geq a_2$, we can use Lemma 39 to conclude that the intertwining operator has non-generic kernel. Therefore $I_{P'}^{GL_N}(\tau'_{\nu'}) = I_{P'}^{GL_N}(Z(\mathcal{S}_1 \cap \mathcal{S}_2) \otimes Z(\mathcal{S}_1 \cup \mathcal{S}_2))$ embeds as a subrepresentations in $I_{P_1}^{GL_N}(\rho(a_1, b_1, a_2, b_2))$ and therefore in $I_P^{GL_N}(St_{n_1}(\rho)|.|^{s_1} \otimes St_{n_2}(\rho)|.|^{s_2})$ by unicity of the generic piece in the induced representation $I_{P_1}^{GL_N}(\rho(a_1, b_1, a_2, b_2))$.

□

We illustrate the theory developed in the previous sections with an example on the general linear group. To lighten notations $|\det|^s$ is simply denoted $|.|^s$.

Example 15. Take $\mathrm{St}_3|.|^5 \times \mathrm{St}_5|.|^1$, a discrete series representation of a Levi subgroup M of GL_8. Notice we take $s_2 = 5 > s_1 = 1$ to be in the Langlands' situation ; i.e to consider the standard module $I_P^{GL_n}(\mathrm{St}_3|.|^5 \times \mathrm{St}_5|.|^1)$.

$$I_P^{GL_n}(\mathrm{St}_3|.|^5 \times \mathrm{St}_5|.|^1) \hookrightarrow I_B^{GL_n}((654)(3210-1))$$

Being a decreasing sequence of integers, the point (654)(3210-1) is a dominant residual point. It is well-known that the twisted Steinberg $\mathrm{St}_8|.|^{5/2}$ is the unique irreducible generic subquotient in the induced module $I_B^{GL_n}((654)(3210-1))$, and appears as subrepresentation.

Consider now the discrete series $St_3|.|^3 \times \mathrm{St}_5|.|^1$ such that

$$I_P^{GL_n}(\mathrm{St}_3|.|^3 \times \mathrm{St}_5|.|^1) \hookrightarrow I_{P_0}^{GL_n}((432)(3210-1))$$

Consider the Weyl group orbit of the point (432)(3210-1), it is clear that there doesn't not exist a decreasing sequence of integers $(\ell(\ell-1)\ldots(\ell-n))$ satisfying the condition for the weighted diagram of type A_n such that a Weyl group element $w \in W$ sends (432)(3210-1) to $(\ell(\ell-1)\ldots(\ell-n))$.

Therefore the point (432)(3210-1) is not a residual point. The unique irreducible generic subquotient will be non-discrete series.

We know it will take the form $I_{P'}^G(\tau'_{\nu'})$ for τ' a generic discrete series of GL_n.

Our focus is on the point (432)(3210-1) : Is there a permutation of this sequence of integers resulting in two different segments $(\ell'\ell'-1\dots(\ell'-m))(\ell(\ell-1)\dots(\ell-p))$ with $m+p=n$, and such that its associated Langlands' parameter ν' is smaller that $\lambda=\nu+s\tilde{a}$?

Consider the sequence (32)(43210-1) ; its associated Langlands' parameter is

$$(5/2,5/2,3/2,3/2,\dots,3/2)$$

whereas $\lambda\equiv(3,3,3,1,\dots 1)$.

$$\lambda-\nu'\equiv(1/2,1/2,3/2,-1/2,\dots,-1/2)$$

As in the proof of Lemma 32, we get $x_1=1/2, x_3=5/2, x_4=2,\dots,x_8=0$. Therefore $\lambda\geq_P\nu'$.

Further analysis allows one to conclude that this is the minimal Langlands' parameter one could obtain.

Therefore the unique irreducible generic subquotient is $I_P^{GL_n}(\mathrm{St}_2|.|^{5/2}\times\mathrm{St}_6|.|^{3/2})$.

Recall now the Bernstein-Zelevinsky theory as in Rodier 1981-1982. In particular the depiction of elementary operations as detailed in Section 5 of Rodier 1981-1982 and the following theorem :

Theorem 73 (Bernstein-Zelevinsky)**.**

(i) The multiplicity of $Z(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_r)$ (the unique subrepresentation in the induced module $I_{P_r}^G(Z(\mathcal{S}_1)\otimes Z(\mathcal{S}_2)\otimes\dots\otimes Z(\mathcal{S}_r))$ is equal to one.

(ii) For a representation $Z(\mathcal{S}'_1,\mathcal{S}'_2,...,\mathcal{S}'_s)$ to be isomorphic to an irreducible subquotient of $I_{P_r}^G(Z(\mathcal{S}_1)\otimes Z(\mathcal{S}_2)\otimes\dots\otimes Z(\mathcal{S}_r))$ it is necessary and sufficient for the set $(\mathcal{S}'_1,\mathcal{S}'_2,...,\mathcal{S}'_s)$ to be constructible from the set $(\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_r)$ by a sequence of elementary operations.

The procedure of taking intersection and union of segments at the root of Bernstein-Zelevinsky's theory coincides with our procedure based on Borel-Wallach's Lemma (Lemma 31).

Indeed on the above example, intersection and union of (432)(3210-1) give precisely (32)(43210-1).

Once the unique irreducible generic subquotient identified, one need to consider the intertwining operator from $I_B^{GL_n}((32)(43210-1))$ to $I_B^{GL_n}((432)(3210-1))$. This operator has non-generic kernel, therefore it sends $I_P^{GL_n}(\mathrm{St}_2|.|^{5/2}\times\mathrm{St}_6|.|^{3/2})$ as a subrepresentation in $I_B^{GL_n}((432)(3210-1))$ and by multiplicity one, as a subrepresentation in $I_P^{GL_n}(\mathrm{St}_3|.|^5\times\mathrm{St}_5|.|^1)$.

D.2. The Generalized Injectivity for SL_n

In [Asgari et Shahidi 2006, Proposition 3.4], Asgari and Shahidi propose the following result :

Proposition 74 (Asgari et Shahidi 2006, Proposition 3.4)**.** *Let $\boldsymbol{G} \subset \widetilde{\boldsymbol{G}}$ be two connected reductive groups whose derived groups are equal. Let $\tilde{P} = \widetilde{\boldsymbol{M}}\boldsymbol{N}$ be a maximal standard parabolic subgroup of $\tilde{\boldsymbol{G}}$ and $\boldsymbol{P}$=$\boldsymbol{MN}$ be the corresponding one in $\boldsymbol{G}$ with $\boldsymbol{M} = \tilde{\boldsymbol{M}} \cap \boldsymbol{G}$.*

Also let $\tilde{\boldsymbol{T}} \subset \tilde{\boldsymbol{M}}$ and $\boldsymbol{T} = \tilde{\boldsymbol{T}} \cap \boldsymbol{G} \subset \boldsymbol{M}$ be maximal tori in $\widetilde{\boldsymbol{G}}$ and $\boldsymbol{G}$, respectively.

Let $\tilde{\tau}$ be a quasi-tempered generic representation of $\tilde{M} = \tilde{M}(F)$ and denote by τ its restriction to $M = \tilde{M}(F)$. Write $\tau = \bigoplus_i \tau_i$ with τ_i irreducible representations of M. The standard module $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ is irreducible if and only if each standard module $I_P^G(\sigma_i)$ is irreducible.

We adapt it to derive the Generalized Conjecture for SL_n once it is proved for GL_n.

Proposition 75. *Let $\boldsymbol{G} \subset \widetilde{\boldsymbol{G}}$ be two connected reductive groups whose derived groups are equal. Let $\tilde{P} = \widetilde{\boldsymbol{M}}\boldsymbol{N}$ be a maximal standard parabolic subgroup of $\tilde{\boldsymbol{G}}$ and $\boldsymbol{P}$=$\boldsymbol{MN}$ be the corresponding one in $\boldsymbol{G}$ with $\boldsymbol{M} = \tilde{\boldsymbol{M}} \cap \boldsymbol{G}$.*

Also let $\tilde{\boldsymbol{T}} \subset \tilde{\boldsymbol{M}}$ and $\boldsymbol{T} = \tilde{\boldsymbol{T}} \cap \boldsymbol{G} \subset \boldsymbol{M}$ be maximal tori in $\widetilde{\boldsymbol{G}}$ and $\boldsymbol{G}$, respectively.

Let $\tilde{\tau}_{s\tilde{\alpha}}$ be a quasi-tempered generic representation of $\tilde{M} = \tilde{M}(F)$ and denote by τ its restriction to $M = \tilde{M}(F)$. Write $\tau = \bigoplus_i \tau_i$ with τ_i irreducible representations of M.

If the unique irreducible generic subquotient of the standard module $I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})$ is a subrepresentation then the unique irreducible generic subquotient of $I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})|_G$ is a subrepresentation.

Proof. Let $I_{\tilde{P}'}^{\tilde{G}}(\tilde{\tau}'_{\nu'})$ be the unique irreducible generic subquotient of the standard module $I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})$. Assume it is a subrepresentation.

$$I_{\tilde{P}'}^{\tilde{G}}(\tilde{\tau}'_{\nu'}) \hookrightarrow I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})$$

Now consider the Restriction Functor to G applied to these two representations, we obtain :

$$I_{\tilde{P}'}^{\tilde{G}}(\tilde{\tau}'_{\nu'})|_G = \bigoplus_i I_P^G(\tau_i) \hookrightarrow I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})|_G \tag{1.23}$$

By uniqueness of Whittaker model for both G and $\tilde{G}$, and observing that since they have the same derived group, they have the same unipotent radical, only one of the $I_P^G(\tau_i)$'s is ψ-generic, where ψ is the character such that the standard module $I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})$ is ψ-generic.

Denote this ψ-generic subquotient $I_P^G(\tau_j)$.

Assume this unique irreducible generic is a subquotient which is not a subrepresentation in $I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})|_G$:

$$I_P^G(\tau_j) \leq I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})|_G$$

Then, $\bigoplus_i I_P^G(\tau_i) \leq I_{\tilde{P}}^{\tilde{G}}(\tilde{\tau}_{s\tilde{\alpha}})|_G$ a contradiction to the above Equation (1.23).

$\square$

Since a generic representation of GL_n breaks up in finitely many representations of SL_n as a direct sum, exactly one of which is generic ; the Proposition 75 above applies to GL_n and SL_n. Therefore, if the generalized injectivity conjecture is proved for GL_n, it follows for SL_n.

D.3. Generalized Injectivity for classical groups

In this subsection we will adapt our results and theorems to the context of classical groups and their variants (see Mœglin 2011 where this terminology was introduced) : In this context, if we let n be the rank of the group $G(n)$, the form of the Levi subgroup M_1 will be assumed isomorphic to $\prod_i GL(k_i) \times G(k_0)$ where the multiset $\{k_0; (k_1, \ldots, k_\ell)\}, n = k_0 + k_1 + \ldots k_\ell, k_0 \geq 0$, index the conjugation classes of Levi subgroup of the group $G(n)$.

Let us detail the conditions for λ to be a residual point for μ^G.

D.4. Conditions for λ to be a residual point for μ^G

In the context of the Subsection 1.5.4.1, where we are choosing σ to be the trivial representation and a character χ_ν of the torus, to observe if the parameter $\lambda \in a_0^*$ is a residual point as defined in the Definition 10, we only have to use the semi- simple rank of G (which is equal to the rank of G if the group is semi-simple).

However, if we are considering σ to be non-trivial ; σ is a representation of a standard Levi subgroup M_1 :

$$M_1 = GL_{k_1} \times \ldots GL_{k_1} \times GL_{k_2} \times \ldots \times GL_{k_2} \times GL_{k_r} \times \ldots \times GL_{k_r} \times G(k)$$

Since rank$(GL_n) = n$ and rank$(SL_n) = n - 1$, the semi-simple rank of GL_{k_i} is $k_i - 1$, then :

$$rk_{ss}(G) = \sum_i k_i d_i + rk_{ss}(G(k))$$

$$rk_{ss}(M_1) = \sum_i (k_i - 1) d_i + rk_{ss}(G(k))$$

The difference is $d = d_1 + d_2 + \ldots + d_r$.

Let us recall that a point λ in $a_{M_1}^*$ is expressed as a r-tuple of λ_i if Σ_σ is the direct sum of r irreducible or empty components $\Sigma_{\sigma,i}$. If each λ_i is a residual point of length d_i, and the sum $d_1 + d_2 + \ldots + d_r$ is equal to d (This is Proposition 24, see its reformulation as Proposition 76 below) as above, λ is a residual point.

As explained in the proof of Proposition 24 (see Proposition 76 below) , assume there is a subsystem $\Sigma_{\sigma,i}$ of type A appearing in Σ_σ, then its associated residual points λ_i is of type $A_{d_i - 1}$; in this case the sum $d_1 + d_2 + d_3 + \ldots d_i - 1$

cannot equals d, and this will insure that λ is not a residual point for the system Σ_σ. Therefore, root subsystems can only be of type B, C, D for λ to be a residual point.

With this observation in mind, let us be more specific.

As in the Section 1.5.3, we first consider the case of an essentially square-integrable representation of a maximal standard Levi subgroup $M \supset M_1$. Here, such representation is a product of a Steinberg representation of $GL_{k_1 \times d_1}$ and an irreducible generic discrete series of $G(k)$. Embedding each of these representations using Heiermann-Opdam result, we have :

$$I_P^G(St_{k_1 \times d_1}(\rho)|.|^s \times \pi) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$$

The Levi subgroup M_1, product of its linear part $M_{1,L}$ and its classical part $M_{1,c}$, is :

$$M_1 = GL_{k_1} \times \ldots GL_{k_1} \times GL_{k_2} \times \ldots \times GL_{k_2} \times GL_{k_r} \times \ldots \times GL_{k_r} \times G(k)$$

The cuspidal support σ_λ can be rewritten :

$$\rho|.|^a \otimes \rho|.|^{a-1} \ldots \rho|.|^b \otimes \underbrace{\sigma_2|.|^{\ell_2} \ldots \sigma_2|.|^{\ell_2}}_{n_{\ell_2,2} \text{ times}} \ldots \underbrace{\sigma_2|.|^0 \ldots \otimes \sigma_2|.|^0}_{n_{0,2} \text{ times}} \ldots$$
$$\underbrace{\sigma_r|.|^{\ell_r} \ldots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \ldots \underbrace{\otimes\sigma_r|.|^0 \ldots \otimes \sigma_r|.|^0}_{n_{0,r} \text{ times}} \otimes \sigma_c$$

In the previous expression, the tuple $(a, \ldots, b)$ is a decreasing sequence of (half)-integers corresponding to a residual segment of type A; whereas for each i, the residual segment is $(\underline{n_i}) := (0, \ldots, 0, n_{\ell_i}, \ldots, n_{1,i}, n_{0,i})$, and ℓ_i is the greatest non-zero (half)-integer in the residual segment $(\underline{n_i})$ (see the Definition 25).

Since by the above reasoning all subsystems $\Sigma_{\sigma,i}$ are of type $B_{d_i}, C_{d_i}, D_{d_i}$ for $i \in \{2, \ldots, r\}$ for ν to constitute a residual point giving π as irreducible generic discrete series, **what remains to consider is the nature of the subsystem $\Sigma_{\sigma,1}$.**

There are clearly two cases, either $\rho \cong \sigma_i$ for one index i, or $\rho \ncong \sigma_i,\ \forall i$. In presenting the proof of the Generalized Injectivity Conjecture in D.4.2 below, we treat separately the two cases.

Let us first state Proposition 24 in the context of classical groups :

Proposition 76. *Assume G is a classical group.*

Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1.

Let

$$d = rk_{ss}(G) - rk_{ss}(M_1).$$

Suppose that

$$M_1 = GL_{k_1} \times \dots GL_{k_1} \times GL_{k_2} \times \dots \times GL_{k_2} \times GL_{k_r} \times \dots \times GL_{k_r} \times G(k)$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G and $\sigma = \sigma_1 \otimes \dots \otimes \sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_2 \dots\dots \otimes \sigma_r \otimes \dots \otimes \sigma_r \otimes \sigma_c$.

Let d_i denote the number of factors equal to σ_i and let $(s_{i,j})_{i,j}$ be a family of non-negative real numbers, $1 \leq i \leq r$, $1 \leq j \leq d_i$ and $s_{i,j} \geq s_{i,j+1}$ for i fixed. Then,

$$\sigma_1|\cdot|^{s_{1,1}} \otimes \dots \sigma_1|\cdot|^{s_{1,d_1}} \otimes \sigma_2|\cdot|^{s_{2,1}} \otimes \dots \sigma_2|\cdot|^{s_{2,d_2}} \otimes \dots \otimes \sigma_r|\cdot|^{s_{r,1}} \otimes \dots \sigma_r|\cdot|^{s_{r,d_r}} \otimes \sigma_c.$$

is in the cuspidal support of a discrete series representations of G, if and only if the following properties are satisfied :

(i) $d = \sum_i d_i$

(ii) one has $\sigma_i \simeq \sigma_i^\vee$ for every i ;

(iii) denote by s_i the unique element in $\{0, 1/2, 1\}$ such that the representation of $G(k+k_i)$ parabolically induced from $\sigma_i|\cdot|^{s_i} \otimes \sigma_c$ is reducible. Then, for all i, $2(s_{i,1}, \dots, s_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of a simple complex adjoint group of

- type D_{d_i} if $s_i = 0$;

- type C_{d_i} if $s_i = 1/2$;

- type B_{d_i} if $s_i = 1$.

Proof. As the group G is semisimple, by Heiermann's result [Theorem 11] $\sigma_{(s_{i,j})_{i,j}}$ is in the cuspidal support of a discrete series representation of G, if and only if it is a residual point of Harish-Chandra's μ-function.

Identify A_{M_1} to $\mathbb{T} = \mathbb{G}_m^{d_1} \times \mathbb{G}_m^{d_1} \times \dots \times \mathbb{G}_m^{d_r}$ and denote $e^{\pm}_{i,j;i',j'}$ the rational character of A_{M_1} identified with $\mathbb{T}$ which sends an element

$$x = (x_{1,1}, \dots, x_{1,d_1}, x_{2,1}, \dots, x_{2,d_2}, \dots, x_{r,1}, \dots, x_{r,d_r})$$

to $x_{i,j}x^{\pm}_{i',j'}$ and by $e^{\pm}_{i,j}$ the one that sends it to $x^{\pm}_{i,j}$.

The μ-function decomposes as $\prod_{\alpha \in \Sigma(P)} \mu^{M_\alpha}$. The function $\lambda \mapsto \mu^{M_\alpha}(\sigma_\lambda)$ won't have a pole or zero on $a^*_{M_1}$ except in the following cases

(i) α is of the form $e^-_{i,j;i,j'}$, $j < j'$;

(ii) $\sigma_i \simeq \sigma_i^\vee$ and α is of the form $e^+_{i,j;i,j'}$, $j < j'$;

(iii) $\sigma_i \simeq \sigma_i^\vee$, $s_i \neq 0$ and α is of the form $e^+_{i,j}$ or $2e^+_{i,j}$ (this depends on the root system).

Denote, the set of these roots $\alpha \in \Sigma(P)$ such that $\lambda \mapsto \mu^{M_\alpha}(\sigma_\lambda)$ has a pole or zero on $a^*_{M_1}$, by Σ_σ.

Let $(t_{i,j})_{i,j}$ be a family of real numbers as in the statement of the proposition and put

$$\sigma_{(t_{i,j})_{i,j}} = \sigma_1|\cdot|^{t_{1,1}} \otimes \dots \sigma_1|\cdot|^{t_{1,d_1}} \otimes \sigma_2|\cdot|^{t_{2,1}} \otimes \dots \sigma_2|\cdot|^{t_{2,d_2}} \otimes \sigma_r|\cdot|^{t_{r,1}} \otimes \dots \sigma_r|\cdot|^{t_{r,d_r}} \otimes \sigma_c.$$

If $\alpha = e^{\pm}_{i,j;i,j'} \in \Sigma_\sigma$, $j < j'$, then

$$\mu^{M_\alpha}(\sigma_{(t_{i,j})_{i,j}}) = c_\alpha(\sigma_{(t_{i,j})_{i,j}})\frac{(1-q^{t_{i,j}\pm t_{i,j'}})(1-q^{-t_{i,j}\mp t_{i,j'}})}{(1-q^{1-t_{i,j}\pm t_{i,j'}})(1-q^{1+t_{i,j}\mp t_{i,j'}})},$$

where $c_\alpha(\sigma_{(t_{i,j})_{i,j}})$ denotes a rational function in $\sigma_{(t_{i,j})_{i,j}}$, which is regular and nonzero for real $t_{i,j}$.

If $\alpha = e_{i,j} \in \Sigma_\sigma$, then

$$\mu^{M_\alpha}(\sigma_{(t_{i,j})_{i,j}}) = c_\alpha(\sigma_{(t_{i,j})_{i,j}})\frac{(1-q^{t_{i,j}})(1-q^{-t_{i,j}})}{(1-q^{\epsilon_\alpha - t_{i,j}})(1-q^{\epsilon_\alpha + t_{i,j}})}$$

with $\epsilon_\alpha = 1, 1/2$.

If $\alpha = 2e_{i,j} \in \Sigma_\sigma$, then

$$\mu^{M_\alpha}(\sigma_{(t_{i,j})_{i,j}}) = c_\alpha(\sigma_{(t_{i,j})_{i,j}})\frac{(1-q^{2t_{i,j}})(1-q^{-2t_{i,j}})}{(1-q^{\epsilon_\alpha - 2t_{i,j}})(1-q^{\epsilon_\alpha + 2t_{i,j}})}$$

with $\epsilon_\alpha = 2, 1$.

Remark that for the last two cases we have used the result of Shahidi on reducibility points for generic cuspidal representations.

Put $\epsilon_i^+ = 0$ if $\sigma_i^\vee \not\simeq \sigma_i$ and put $\epsilon_i = 0$ if $\sigma_i^\vee \not\simeq \sigma_i$ or $s_i = 0$. It follows that, for $\sigma_{(s_{i,j})_{i,j}}$ to be a residual point of Harish-Chandra's μ-function, it is necessary and sufficient, that for every i, one has

$$\begin{aligned} d_i = {}& |\{(j,j')|j<j', s_{i,j}-s_{i,j'}=1\}| + \epsilon_i^+|\{(j,j')|j<j', s_{i,j}+s_{i,j'}=1\}| + \epsilon_i|\{j|s_{i,j}=\epsilon_{e_{i,j}}\}| \\ & -2[|\{(j,j')|j<j', s_{i,j}-s_{i,j'}=0\}| + \epsilon_i^+|\{(j,j')|j<j', s_{i,j}+s_{i,j'}=0\}| + \epsilon_i|\{j|s_{i,j}=0\}|]. \end{aligned}$$

Remark first that, if $\sigma_i^\vee \not\simeq \sigma_i$, then in the above only roots which form a root system of type A_{d-1} are involved, but by 5.6.1 in Carter 1985 the number above can then at most be equal to $d_i - 1$. If $\epsilon_{e_{i,j}} = 1$ or $s_i = 0$, then this is the condition for $2(s_{i,1}, \ldots, s_{i,d_i})$ defining a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type B_{d_i}, C_{d_i} or D_{d_i} as in [5.7.5 in Carter 1985], and one has $s_i = 1$ for B_{d_i} and $s_i = 1/2$ for C_{d_i}. If $\epsilon_{e_{i,j}} = 1/2$, then the root system is of type B_{d_i}, $s_i = 1/2$ and, after multiplying the short roots by 2, one sees that $2(s_{i,1}, \ldots, s_{i,d_i})$ defines a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type C_{d_i}. If $\epsilon_{e_{i,j}} = 2$, then the root system is of type C_{d_i}, $s_i = 1$ and, after dividing the long roots by 2, one sees that $2(s_{i,1}, \ldots, s_{i,d_i})$ defines a nilpotent element in the Lie algebra of an adjoint simple complex group of type B_{d_i}.

In other words, $2(s_{i,1}, \ldots, s_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of an adjoint simple complex group of type B_n, C_n or D_n, if s_i is respectively 1, 1/2 or 0. □

Proposition 77. *Let us consider an irreducible generic cuspidal representation σ_λ of M_1.*

With the context of the previous paragraph, it can be written :

$$\sigma_\lambda := \rho|.|^{a} \otimes \rho|.|^{a-1} \dots \rho|.|^{b} \otimes \underbrace{\sigma_2|.|^{\ell_2} \dots \sigma_2|.|^{\ell_2}}_{n_{\ell_2,2} \text{ times}} \dots \underbrace{\sigma_2|.|^{0} \dots \otimes \sigma_2|.|^{0}}_{n_{0,2} \text{ times}} \dots$$
$$\underbrace{\sigma_r|.|^{\ell_r} \dots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \dots \underbrace{\otimes\sigma_r|.|^{0} \dots \otimes \sigma_r|.|^{0}}_{n_{0,r} \text{ times}} \otimes\sigma_c$$

Let us assume the representation σ_λ is in the cuspidal support of the standard module $I_P^G(St(\rho)|.|^s \times \pi)$.

Take $\rho \cong \rho^\vee$ to satisfy the hypothesis of Proposition 24 and further assume $\rho \ncong \sigma_i$ for any $i \in \{2, r\}$.

As in Proposition 76, we consider the reducibility points of $I_{P_1}^{G(k+k_1)}(\rho|det_{k_1}|^{s_1}\otimes\sigma_c)$.

Then λ is a residual point, which is equivalent by Theorem 11 to say the generic induced representation $I_{P_1}^G(\sigma_\lambda)$ has a generic discrete series subquotient if and only if :

1. *$s_1 = 0$ and the residual segment $(a, \dots, b)$ has to be $(d_1 - 1, \dots, 2, 1, 0)$, then $s = \frac{d_1-1}{2}$.*
2. *$s_1 = 1/2$, and the residual segment $(a, \dots, b)$ has to be $(\frac{2d_1-1}{2}, \frac{2d_1-1}{2} - 1, \dots, \frac{1}{2})$, then $s = \frac{a+b}{2} = d_1$.*
3. *$s_1 = 1$, and the residual segment $(a, \dots, b)$ has to be $(d_1, \dots, 2, 1)$ then $s = \frac{d_1-1}{2}$.*

Proof. For λ to constitute a residual point, we need λ_1 to constitute a residual point of type B_{d_1}, C_{d_1} or D_{d_1} so that the condition $d_1 + d_2 + \dots d_r = d$ is satisfied.

Notice that $\lambda_1 = (a, a-1, \dots, b)$ is a strictly decreasing sequence of (half)-integers of length d_1 ; for it to correspond to a dominant residual point of type B_{d_1}, C_{d_1} or D_{d_1}, we need to consider some specific residual segments given by :

1. $s_1 = 0$, the root system to consider is D_{d_1} and the residual segment $(a, \dots, b)$ has to be $(d_1 - 1, \dots, 2, 1, 0)$, then $s = \frac{d_1-1}{2}$.
2. $s_1 = 1/2$, the root system to consider is C_{d_1} and the residual segment $(a, \dots, b)$ $(\frac{2d_1-1}{2}, \frac{2d_1-1}{2} - 1, \dots, \frac{1}{2})$, then $s = \frac{a+b}{2} = d_1$
3. $s_1 = 1$, the root system to consider is B_{d_1} and the residual segment $(a, \dots, b)$ has to be $(d_1, \dots, 2, 1)$ then $s = \frac{d_1-1}{2}$.

□

Let us assume now that $\rho \cong \sigma_i$ for some $i \in \{2, r\}$.

Let us recall (as in Example 8) our previous observations on intertwining operators with non-generic kernels : we fix an irreducible cuspidal representation of a linear group ρ and two cuspidal strings $(a, b, \underline{n})$ and $(a', b', \underline{n'})$ in the same W_σ-cuspidal string ; i.e, there exists a Weyl group element in W_σ such that $w(a, b, \underline{n}) = (a', b', \underline{n'})$.

We examine the intertwining operator between the *principal series* $I_{P_1}^G(\rho(a, b, \underline{n}) \otimes \sigma_c)$ and $I_{P_1}^G(\rho(a', b', \underline{n'}) \otimes \sigma_c)$. As explained in Section 1.3, Proposition 18, this operator can be decomposed in rank one operators. Let us repeat how we assess the non- genericity of the kernel of rank one operators (see Example 8 for details) :

For any two adjacent elements in the residual segments : $\{a, b\}$, the rank one operator goes from $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\rho|.|^a \otimes \rho|.|^b)$ to $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\rho|.|^b \otimes \rho|.|^a)$, with s_α the transposition from (a, b) to (b, a). Therefore it is an operator with non-generic kernel if and only if $a < b$; indeed if $\lambda := (a, b)$, then $\langle \check{\alpha}, \lambda \rangle = a - b < 0$.

Further if the Weyl group W_σ is isomorphic to $S_n \rtimes \{\pm 1\}$, the Weyl group element corresponding to $\{\pm 1\}$ is the sign change for the extreme right element of the cuspidal string : then the operator $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\rho|.|^{-a} \otimes \sigma_c)$ to $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\rho|.|^a \otimes \sigma_c)$ has non-generic kernel.

Proposition 78. *Let us keep the context of the previous Proposition 77 but assume $\rho \cong \sigma_i$ for some $i \in \{2, r\}$.*

Then λ is a residual point if and only if $(a, \dots, b)(\underline{n_i})$ is a residual point. This means there exists a Weyl group element w in W_σ such that $w(a, \dots, b)(\underline{n_i}) = (\underline{n_i'})$ where the residual segment $(\underline{n_i})$ and $(\underline{n_i'})$ are of the same type.

Proof. Let us recall the expression of the representation σ_λ of M_1 :

$$\begin{aligned}\sigma_\lambda = \rho|.|^a \otimes \rho|.|^{a-1} \dots \rho|.|^b \otimes \underbrace{\sigma_2|.|^{\ell_2} \dots \sigma_2|.|^{\ell_2}}_{n_{\ell_2,2} \text{ times}} \dots \underbrace{\sigma_2|.|^0 \dots \otimes \sigma_2|.|^0}_{n_{0,2} \text{ times}} \dots \\ \underbrace{\sigma_r|.|^{\ell_r} \dots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \dots \underbrace{\otimes \sigma_r|.|^0 \dots \otimes \sigma_r|.|^0}_{n_{0,r} \text{ times}} \otimes \sigma_c \quad (1.24)\end{aligned}$$

Suppose we reorganize the form of the representation σ_λ by permuting and changing sign on the elements in the residual segments corresponding to the two isomorphic components ρ and σ_i. Then, the intertwining operators between the induced modules before and following such operations may have generic kernel.

It is enough to consider only those intertwining operators since other rank one operators between modules induced from two representations in disjoint inertial orbits are one-to-one, as explained in the Proposition 71.

Without loss of generality, we can therefore assume $\rho \cong \sigma_r$. If there exists a Weyl group element w in W such that $w(a, \dots, b)(\underline{n_r}) = (\underline{n_r'})$ where the residual segment $(\underline{n_r})$ and $(\underline{n_r'})$ are of the same type, we use intertwining operators to reorganize the cuspidal support so as to obtain a dominant residual point λ_r'. The form of λ is $(\lambda_2, \lambda_3, \dots, \lambda_r')$.

Let us describe the used intertwining operators :

First, the genericity of the kernel of the rank one operator from $I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\rho|.|^a\otimes\sigma_i|.|^b)$ to $I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\sigma_i|.|^b\otimes\rho|.|^a)$ is treated in the paragraph just above the Proposition.

Regarding the intertwining operator from $I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\rho|.|^{-a}\otimes\sigma_c)$ to $I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\rho|.|^a\otimes\sigma_c)$, we proceed in three steps :

First, there is a sequence of one-to-one operators between

$$I^G_{P_1}(\ldots\otimes\rho|.|^{-a}\otimes\sigma_2|.|^{\ell_2}\otimes\ldots\otimes\sigma_r|.|^{\ell_r}\otimes\ldots\sigma_r|.|^0\otimes\otimes\sigma_c)$$

to

$$I^G_{P_1}(\ldots\otimes\sigma_2|.|^{\ell_2}\otimes\ldots\otimes\rho|.|^{-a}\otimes\sigma_r|.|^{\ell_r}\otimes\ldots\sigma_r|.|^0\otimes\otimes\sigma_c)$$

Secondly, there is a composition of rank one operators with non-generic kernels from

$$I^G_{P_1}(\ldots\rho|.|^{-a}\otimes\sigma_r|.|^{\ell_r}\otimes\sigma_r|.|^{\ell_r-1}\otimes\ldots\sigma_r|.|^0\otimes\sigma_c)$$

to

$$I^G_{P_1}(\ldots\otimes\sigma_r|.|^{\ell_r}\otimes\sigma_r|.|^{\ell_r-1}\otimes\ldots\sigma_r|.|^0\otimes\rho|.|^{-a}\otimes\sigma_c)$$

Finally, we can apply the rank one operator with non-generic kernel from

$$I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\rho|.|^{-a}\otimes\sigma_c)\qquad\text{to}\qquad I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\rho|.|^a\otimes\sigma_c)$$

Eventually, applying the inverse of the one-to-one operators, we get to

$$I^G_{P_1}(\sigma_2|.|^{\ell_2}\otimes\sigma_2|.|^{\ell_2-1}\otimes\ldots\sigma_2|.|^0\otimes\ldots\rho|.|^a\ldots)$$

There the exponent a is positioned appropriately in the sequence $(\ell_r,\ldots,0)$ such that this sequence remains a decreasing sequence of (half)-integers. $\square$

From these considerations, it becomes clear that the cuspidal representation of a linear group ρ possibly isomorphic to some σ_r, and the cuspidal string $(a, b, \underline{n_r})$ will be the focal point to study. It is somehow the « DNA molecule » characterizing the unique generic subquotient in a standard module. Its composition and its Jordan block will allow us to conduct all our argumentation.

D.4.1. Moeglin's Lemma and the Embedding Result

Let $\mathcal{S}=[\rho,\rho(1),\ldots,\rho(r-1)]$ be a segment as defined in Rodier 1981-1982 and recall $Z(\mathcal{S})$ is the unique essentially square- integrable subrepresentation in the induced module $\rho\times\ldots\times\rho(r-1)$. Often, we denote it $Z(\rho,r-1,0)$, and more generally $Z(\rho,a,b)$ for a and b any two (half)-integers.

Then, let us state Moeglin's Lemma as originally stated in Moeglin 2002 :

Lemma 79 (Moeglin, Section 5 in MOEGLIN 2002)**.** *Let π_0 be an irreducible discrete series representation of a classical group G and let a and a_- be two consecutive elements in the Jordan block of π_0. Let us assume there exists an irreducible representation π' of $G(n - d_\rho(a - a_-)/2, F)$ for d_ρ the dimension of an irreducible autodual cuspidal representation of GL, and an embedding : $\pi_0 \hookrightarrow Z(\rho, \frac{a-1}{2}, \frac{a_- - 1}{2} + 1) \times \pi'$; then there exists a discrete series π_0' such that :*

$$\pi_0 \hookrightarrow Z(\rho, \frac{a-1}{2}, -\frac{a_- - 1}{2}) \times \pi_0'$$

The reader has noticed that since we are in the context of classical groups we use the terminology *Jordan block*.

We aim to prove a certain embedding result [Proposition 3.1 in HANZER 2010] based on the knowledge of Heiermann-Opdam Result [Proposition 12]. In the context of any quasi-split reductive group it was Proposition 56 above, in this section it is Proposition 81 below.

Illustrating the same methods of proof, we first present an auxiliary result which might be of interest for classical groups specialists :

Proposition 80. *Let $(\underline{n})$ be a residual segment corresponding to a dominant residual point and let $2a_1 + 1 > 2a_2 + 1 > \ldots > 2a_n + 1$ be Jordan block elements of the associated irreducible generic discrete series of a classical group (or its variant) G : π_0, i.e there exists irreducible generic cuspidal representations of a general linear group ρ (self-dual) and σ_c a cuspidal representation of the same classical group of smaller rank such that $\pi_0 \hookrightarrow I_{P_1}^G(\rho(\underline{n}) \otimes \sigma_c)$.*

Then $\pi_0 \hookrightarrow Z(\rho, \ell+m = a_1, \epsilon_1) \times Z(\rho, a_2, \epsilon_2) \times \ldots \times Z(\rho, a_n, \epsilon_n) \times \underbrace{\rho\nu^1}_{x \text{ times}} \times \underbrace{\rho\nu^0}_{y \text{ times}} \rtimes \sigma_c.$

The ϵ_i are positive (half)-integers as small as possible given the constraints of the tuple $\underline{n}$, and therefore x and y are possibly null.

Proof. We fix ρ and therefore we will omit it in the following argumentation, e.g. $I_{P_1}^G(\rho(\underline{n}))$ will be simply denoted $I_{P_1}^G((\underline{n}))$ whereas $\nu^i\rho$ will be ν^i, where ν classically denotes $|\det|$. Without loss of generality, we conduct the proof with a segment of integers, the same proof holds for a segments of half-integers.

To develop the argumentation in this proof, it will also be convenient to use to following notation :

$$I_{P_1}^G(\underline{n}) := \nu^{\ell+m} \times \ldots \times \nu^{\ell} \times \underbrace{\nu^{\ell-1}}_{2 \text{ times}} \times \underbrace{\nu^{\ell-2}}_{n_{\ell_2} \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c$$

Using the fact that the segment $(\ell - 1)$ is included in $(\ell + m, \ldots, \ell - 1)$, hence they are unlinked, Bernstein-Zelevinsky's Theorem implies

$$Z(\ell + m, \ell - 1) \times \nu^{\ell-1} \cong \nu^{\ell-1} \times Z(\ell + m, \ell - 1)$$

$$\nu^{\ell-1} \times Z(\ell+m,\ell-1) \underbrace{\nu^{\ell-2}}_{n_{\ell-2}\text{ times}} \times \ldots \times \underbrace{\nu^{0}}_{n_0\text{ times}} \rtimes \sigma_c$$
$$\hookrightarrow \nu^{\ell-1} \times \nu^{\ell+m} \times \ldots \times \nu^{\ell} \times \underbrace{\nu^{\ell-1}}_{2\text{ times}} \times \underbrace{\nu^{\ell-2}}_{n_{\ell_2}\text{ times}} \times \ldots \times \underbrace{\nu^{0}}_{n_0\text{ times}} \rtimes \sigma_c = \Pi_1 \quad (1.25)$$

But in the right hand induced representation Π_1 space, we also have a subrepresentation $\nu^{\ell-1} \times Z(\ell+m,\ldots,\ell-1,\ell-2) \times \nu^{\ell-2} \times \ldots \times \sigma_c = \Xi_1$. Let's assume that $\pi_0 \not\hookrightarrow \Xi_1 = \nu^{\ell-1} \times Z(\ell+m,\ldots,\ell-1,\ell-2) \times \nu^{\ell-2} \times \ldots \times \sigma_c$ Then since Π_1 is a length two representation, we have :

$$\pi_0 \hookrightarrow \Pi_1/\nu^{\ell-1} \times Z(\ell+m,\ldots,\ell-1,\ell-2) \times \nu^{\ell-2} \times \ldots \times \sigma_c$$
$$\cong \nu^{\ell-1} \times L(\ell+m,\ldots,\ell-1,\ell-2) \times \nu^{\ell-2} \times \ldots \times \sigma_c = \Sigma_1$$

$L(\ell+m,\ldots,\ell-1,\ell-2)$ classically denotes the Langlands quotient of the induced representation $\nu^{\ell+m} \times \nu^{\ell-1} \times \ldots \times \nu^{\ell-2}$; similarly Langlands' quotient of $Z(a_1, b_1) \times Z(a_2, b_2)$ would be denoted $L(Z(a_1, b_1), Z(a_2, b_2))$.

By GL_n theory, the representation $Z(\ell+m,\ldots,\ell-1,\ell-2)$ is generic and therefore Ξ_1 has a unique irreducible generic subquotient by Rodier's theorem. We may call it π'. Then Ξ_1 and Σ_1 contain irreducible generic subquotients π_0 and π'_0 and Π_1 would have two irreducible generic subquotients, contradicting Rodier's theorem. Therefore

$$\pi_0 \hookrightarrow \Xi_1 = \nu^{\ell-1} \times Z(\ell+m,\ldots,\ell-1,\ell-2) \times \nu^{\ell-2} \times \ldots \times \sigma_c$$

It is also possible to use the standard module conjecture $\nu^{\ell-1} \times L(\ell+m,\ldots,\ell-1,\ell-2) \times \nu^{\ell-2} \times \ldots \times \sigma_c$ cannot be generic else, $L(\ell+m,\ldots,\ell-1,\ell-2) \cong \nu^{\ell+m} \times \nu^{\ell-1} \times \ldots \times \nu^{\ell-2}$ but the later was assumed reducible.

Further this is equivalent to say $\pi_0 \hookrightarrow \nu^{\ell-1} \times \nu^{\ell-2} \times Z(\ell+m,\ell-1,\ell-2) \times \underbrace{\nu^{\ell-3}}_{n_{\ell-3}-1\text{ times}} \times \ldots \times \sigma_c$ and by the exact same argumentation developed above, we can further write :

$$\pi_0 \hookrightarrow \nu^{\ell-1} \times \nu^{\ell-2} \times Z(\ell+m,\ell-3) \times \nu^{\ell-3} \times \ldots \times \sigma_c \cong \nu^{\ell-1} \times \nu^{\ell-2} \times \nu^{\ell-3} \times Z(\ell+m,\ell-3) \times \nu^{\ell-3} \times \ldots \times \sigma_c$$

Repeating this procedure, we get :

$$\pi_0 \hookrightarrow \nu^{\ell-1} \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2}-1\text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1-1\text{ times}} \times \underbrace{\nu^{0}}_{n_0-1\text{ times}} \times Z(\ell+m,0) \rtimes \sigma_c$$

On the piece $\nu^{\ell-1} \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2}-1 \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1-1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0-1 \text{ times}}$, one repeats the same procedure to embed the unique irreducible generic subquotient as subrepresentation as follows :

$$\pi_0 \hookrightarrow \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2}-2 \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1-2 \text{ times}} \times \underbrace{\nu^{0}}_{n_0-2 \text{ times}} \times Z(\ell-1,0) \times Z(\ell+m,0) \times \sigma_c$$

Eventually, we embed the unique irreducible generic discrete series : π_0 as subrepresentation as follows : $\pi_0 \hookrightarrow Z(\ell+m = a_1, \epsilon_1) \times Z(a_2, \epsilon_2) \times \ldots \times Z(a_n, \epsilon_n) \times \underbrace{\nu^{1}}_{x \text{ times}} \times \underbrace{\nu^{0}}_{y \text{ times}} \rtimes \sigma_c$. □

Proposition 81. *Let $(\underline{n})$ be a residual segment corresponding to a dominant residual point and let $2a_1+1 > 2a_2+1 > \ldots > 2a_n+1$ be Jordan block elements of an irreducible generic discrete series of a classical group (or its variant) G : π_0, i.e there exists an irreducible generic cuspidal representations of a general linear group ρ (self-dual) and σ_c a cuspidal representation of the same classical group of smaller rank such that $\pi_0 \hookrightarrow I_{P_1}^G(\rho(\underline{n}) \otimes \sigma_c)$.*

For any i, there exists an irreducible generic discrete series τ' (to which corresponds the residual segment $(\underline{n'})$) and an embedding as a subrepresentation of the unique irreducible generic subquotient in

$$Z(\rho, a_i, -a_{i+1}) \times \tau' \hookrightarrow I_{P_1}^G(\rho(a_i, \ldots, -a_{i+1})(\underline{n'}))$$

Where the residual segment $(a_i, \ldots, -a_{i+1})(\underline{n'})$ is in the W_σ orbit of the residual segment $(\underline{n})$.

Proof. We fix ρ and therefore we will omit it in the following argumentation, e.g. $I_{P_1}^G(\rho(\underline{n}))$ will be simply denoted $I_{P_1}^G(\underline{n})$ whereas $\nu^i\rho$ will be ν^i, where ν classically denotes $|\det|$. Without loss of generality, we conduct the proof with a segment of integers, the same proof holds for a segment of half-integers.

By Heiermann-Opdam's Result [Proposition 12] and Lemma 34, to any residual segment $(\underline{n})$ we associate the unique irreducible generic discrete series subquotient in $I_{P_1}^G(\underline{n}) := \nu^{\ell+m} \times \ldots \times \nu^{\ell} \times \underbrace{\nu^{\ell-1}}_{2 \text{ times}} \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2} \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c$

Then as explained in the subsection 1.5.2 this segment defines uniquely Jordan block elements $2a_1+1 > 2a_2+1 > \ldots > 2a_n+1$.

Start with the two elements $a_1 = \ell+m$ and $a_2 = \ell-1$.

From $I_{P_1}^G(\underline{n}) = I_{P_1}^G(\ell+m, a_2 = \ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots))$ it is clear that $Z(a_1, a_2+1 = \ell) \times \underbrace{\nu^{\ell-1}}_{2 \text{ times}} \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2} \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c \hookrightarrow I_{P_1}^G(\underline{n})$

Now consider the two consecutive elements $a_2 = \ell-1$ and a_3 ; The representation $\Xi_1 := \nu^{\ell-1} \times Z(a_1, \ell-1) \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2} \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c \hookrightarrow \nu^{\ell-1} \times$

$$\nu^{\ell+m} \times \times \ldots \underbrace{\nu^{\ell-2}}_{n_{\ell-2} \text{ times}} \times \ldots \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c = \Pi_1$$

Using GL theory, the representation Ξ_1 is generic and therefore Ξ_1 has a unique irreducible generic subquotient by Rodier's theorem. If π_0 does not embed in Ξ_1 then it embeds in Π_1/Ξ_1. Further, the unique irreducible generic subquotient of Ξ_1 we denote π_0'. But then Π_1 would contain π_0 and π_0' contradicting Rodier's theorem. Therefore

$$\pi_0 \hookrightarrow \Xi_1$$

Repeating this procedure we come to : $\pi_0 \hookrightarrow \nu^{\ell-1=a_2} \times \nu^{\ell-2} \times \nu^{\ell-3} \times \ldots \times \nu^{a_3+1} \times Z(a_1, \ell-1) \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2}-1 \text{ times}} \times \ldots \times \underbrace{\nu^{a_3+1}}_{n_{a_3+1}-1 \text{ times}} \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c$

And therefore :

$$\pi_0 \hookrightarrow Z(a_2, a_3+1) \times Z(a_1, \ell-1) \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2}-1 \text{ times}} \times \ldots \times \underbrace{\nu^{a_3+1}}_{n_{a_3+1}-1 \text{ times}} \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c$$

Yet there is left to show that there exists an irreducible representation θ_2 such that $\pi_0 \hookrightarrow Z(a_2, a_3+1) \times \theta_2$. We will use a filtration of the representation of the representation $\Theta_2 := Z(a_1, \ell-1) \times \underbrace{\nu^{\ell-2}}_{n_{\ell-2}-1 \text{ times}} \times \ldots \times \underbrace{\nu^{a_3+1}}_{n_{a_3+1}-1 \text{ times}} \times \underbrace{\nu^{1}}_{n_1 \text{ times}} \times \underbrace{\nu^{0}}_{n_0 \text{ times}} \rtimes \sigma_c$ to obtain the irreducible representation θ_2.

We can write a filtration of generic pieces of the representation space Θ_2 : $V_0 = \{0\} \subseteq V_1 \subseteq V_2 \subseteq \ldots$ where V_{i+1}/V_i is irreducible for any i. Assume

$$\pi_0 \not\hookrightarrow Z(a_2, a_3+1) \times V_1 \tag{1.26}$$

then $\pi_0 \hookrightarrow Z(a_2, a_3+1) \times \Theta/V_1$ and $V_1/V_2 \subseteq \Theta/V_1$

If

$$\pi_0 \not\hookrightarrow Z(a_2, a_3+1) \times V_2/V_1 \tag{1.27}$$

then from (1.26) and (1.27)

$$\pi_0 \not\hookrightarrow Z(a_2, a_3+1) \times V_2$$

Therefore $\pi_0 \hookrightarrow \; Z(a_2, a_3+1) \times \Theta/V_2$; repeating this procedure we eventually need to have some index j so that

$$\pi_0 \hookrightarrow Z(a_2, a_3+1) \times V_{j+1}/V_j$$

Eventually, using Moeglin's Lemma, we can conclude that there exists a generic discrete series τ_2 such that

$$\pi_0 \hookrightarrow Z(a_2, -a_3) \times \tau_2$$

For any two consecutive elements in the Jordan block $2a_i + 1$ and $2a_{i+1} + 1$, the same argumentation (i.e first embedding π_0 as a subrepresentation in $Z(a_i, a_{i+1} + 1) \times \theta_i$, and conclude with Moeglin's Lemma 79), yields the embedding : $\pi_0 \hookrightarrow Z(a_i, -a_{i+1}) \times \tau_i$ for a generic discrete series τ_i. $\square$

Remark 10. By our choice of a_i and a_{i+1}, the segment $(\rho|.|^{a_i}, \ldots, \rho|.|^{-a_{i+1}})$ gives a generic discrete subrepresentation of a linear group, $St_{n_i}(\rho)|.|^{s_{a_i}}$, where $s_{a_i} = \frac{a_i - a_{i+1}}{2}$ is the first reducibility point of the induced representation $I_{P_1}^G(St_{n_i}(\rho)|.|^{s_{a_i}} \times \tau_i)$.

Corollary 2 (of Propositions 80 and 81)**.** *Let $\sigma_\lambda := \bigotimes_i \sigma_i(\underline{n_i})$ be the cuspidal support of an irreducible generic discrete series of a classical group (or its variant) : π_0.*

Isolate a segment $(\underline{n_j}) := (\underline{n})$ (typically, if π_0 is the unique irreducible generic subquotient in a standard module $Z(\rho, \mathfrak{a}, \mathfrak{b}) \times \tau$, then we isolate the segment associated to the cuspidal representation of linear group ρ). Let $2a_1+1 > 2a_2+1 > \ldots > 2a_n+1$ be Jordan block elements of the segment $(\underline{n})$.

Then π_0 embeds as a subrepresentation as follows :

1. $\pi_0 \hookrightarrow Z(\rho, \ell+m = a_1, \epsilon_1) \times Z(\rho, a_2, \epsilon_2) \times \ldots \times Z(\rho, a_n, \epsilon_n) \times \underbrace{\rho\nu^1}_{x \text{ times}} \times \underbrace{\rho\nu^0}_{y \text{ times}} \rtimes \tau_0$
$$\hookrightarrow I_{P_1}^G(\rho(\ell+m = a_1, \epsilon_1) \otimes (a_2, \epsilon_2) \otimes \ldots \otimes (a_n, \epsilon_n) \bigotimes_{i \neq j} \sigma_i(\underline{n_i}) \otimes \sigma_c).$$
The ϵ_i are positive (half)-integers as small as possible given the constraints of the tuple $\underline{n}$, and therefore x and y are possibly null.
2. *For any i, there exists a irreducible generic discrete series τ' (to which corresponds the residual segment $(\underline{n'})$) and an embedding as a subrepresentation of the unique irreducible generic subquotient π_0 in*
$$Z(\rho, a_i, -a_{i+1}) \times \tau' \hookrightarrow I_{P_1}^G(\rho((a_i, \ldots, -a_{i+1}) + (\underline{n'})) \otimes \sigma_i(\underline{n_i}) \otimes \sigma_c).$$
Where the residual segment $(a_i, \ldots, -a_{i+1})(\underline{n'})$ is in the W_σ orbit of the residual segment $(\underline{n})$.

Proof. By Proposition 12 and Lemma 34, we have $\pi_0 \hookrightarrow I_{P_1}^G(\rho(\underline{n}) \otimes \bigotimes_i \sigma_i(\underline{n_i}) \otimes \sigma_c)$. The proofs of Propositions 80 and 81 can be reproduced. $\square$

D.4.2. The proof

Using this result, we prove the generalized injectivity conjecture for discrete series subquotient.

We consider the case of a standard module induced from a maximal parabolic subgroup in Proposition 82, and then extend the result to non-necessarily maximal standard parabolic subgroups in Theorems 83 and 84.

Proposition 82. *Let $\rho(\mathfrak{a}, \ldots, \mathfrak{b}) \bigotimes_i \sigma_i(\underline{n_i}) \otimes \sigma_c$ be in the cuspidal support of a generic essentially square- integrable representation $St_{t_\mathfrak{b}}(\rho)|.|^{s_\mathfrak{b}} \otimes \pi$ of a maximal Levi subgroup M of a classical group.*

Assume $\sigma_j \cong \rho$ for some j and we take $\rho(\underline{n'_j}) \otimes (\sigma_i(\underline{n_i}))_{i \neq j} \otimes \sigma_c$ a dominant residual point in the conjugacy class of the cuspidal support. This is equivalent to say that the induced representation

$$I_{P_1}^G(\rho(a, \ldots, b); (\sigma_i(\underline{n_i}))_i; \sigma_c)$$

has a discrete series subquotient.

Given such a, we show there exists a_- such that $-a_- \leq b$ and $(2a+1, 2a_-+1)$ are consecutive elements in a Jordan block and therefore satisfy the conditions of Corollary 2.

Then, this unique irreducible generic subquotient which is discrete series embeds as a submodule in $I_{P_1}^G(\rho(a, b); (\sigma_i(\underline{n_i}))_i, \sigma_c)$ and therefore in the standard module

$$I_P^G(St_{t_b}(\rho)|.|^{s_b} \otimes \pi) \hookrightarrow I_{P_1}^G(\rho(a, b); \sigma_i(\underline{n_i})_i, \sigma_c)$$

Proof. Let γ be an irreducible generic discrete series representation which appears as subquotient in a standard module $I_P^G(St_{t_b}|.|^{s_b} \otimes \pi)$ induced from a maximal standard parabolic subgroup P of G. Let $\rho(a, \ldots, b) \bigotimes_i (\sigma_i(\underline{n_i}) \otimes \sigma_c)$ be its cuspidal support.

Letting j be the unique index such that $\sigma_j \cong \rho$, without loss of generality we can assume $j = r$, i.e $\sigma_r \cong \rho$. Indeed, the result of Proposition 71 claimed that elementary intertwining operators interchanging cuspidal representations in two disjoint inertial orbits are one-to-one. We can therefore reorganize the cuspidal support to have the unique index j such that $\sigma_j \cong \rho$ in the last position.

Further, using these elementary intertwining operators, we reorganize the cuspidal support such that the residual segment $\rho(a, b)$ is next to the residual segment $\sigma_r(\underline{n_r})$; i.e :

$$\begin{aligned}\sigma_\lambda := & \underbrace{\sigma_2|.|^{\ell_2} \ldots \sigma_2|.|^{\ell_2}}_{n_{\ell_2,2} \text{ times}} \ldots \underbrace{\sigma_2|.|^{0} \ldots \otimes \sigma_2|.|^{0}}_{n_{0,2} \text{ times}} \\ & \ldots \otimes \rho|.|^{a} \otimes \rho|.|^{a-1} \ldots \rho|.|^{b} \otimes \underbrace{\sigma_r|.|^{\ell_r} \ldots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \ldots \underbrace{\otimes \sigma_r|.|^{0} \ldots \otimes \sigma_r|.|^{0}}_{n_{0,r} \text{ times}} \otimes \sigma_c \qquad (1.28)\end{aligned}$$

By Proposition 12, there exists a parabolic subgroup P' such that π_0 embeds as a subrepresentation in the induced module $I_{P'}^G(\sigma'_{\lambda'})$, for λ' a dominant residual point.

Let $(w\sigma)_{w\lambda}$ be the dominant (for P_1) residual point in the Weyl group W_σ orbit of σ_λ, then (using Theorem 2.9 in Bernstein et Zelevinsky 1977 or Theorem VI.5.4 in Renard 2010) π_0 is the unique irreducible generic subquotient in $I_{P_1}^G((w\sigma)_{w\lambda})$, $w\lambda$ dominant for P_1 and Proposition 16 gives us that these two are isomorphic.

The point $(w\sigma)_{w\lambda} \cong \sigma_{w\lambda}$ is a dominant residual point with respect to P_1 : $w\lambda \in \overline{a_{M_1}^*}^+$ and there is a unique element in the Weyl group orbit of a residual point which is dominant and is explicitely given by a residual segment using the correspondence of the Subsection 1.5.1. Denote $w\lambda := (w\lambda_1, w\lambda_2, \dots, w\lambda_r)$ this dominant residual point, and we set now $\Sigma_{\mathcal{O},\mu,1} = \emptyset$ and $w\lambda_r = (\underline{n'_r})$.

We therefore have :

$$\gamma \hookrightarrow I_{P_1}^G(\rho(\underline{n'_r}) \bigotimes_{i\neq r} \sigma_i(\underline{n_i}) \otimes \sigma_c)$$

Assume $2a+1$ and $2a_- +1$ are consecutive elements in a Jordan block and satisfy the conditions of Corollary 2.

Let π_{-a_-} be the generic discrete series representation with cuspidal support $\bigotimes_{i\neq r} \sigma_i(\underline{n_i}) \otimes \rho(\underline{n_{a_-}})$ associated to the residual segment $(\underline{n_{-a_-}})$.

Let $s_{-a_-} = \frac{a-a_-}{2}$ and t_{-a_-} be the length of the segment $(a, \dots, -a_-)$. By Corollary 2, we can write

$$\gamma \hookrightarrow I_P^G(St_{t_{a_-}}(\rho)|.|^{s_{a_-}} \otimes \pi_{a_-}) \hookrightarrow I_{P_1}^G(\bigotimes_{i\neq r} \sigma_i(\underline{n_i}) \otimes \rho(a, \dots, -a_-, \underline{n_{a_-}}))$$

Here, we need to justify that given a, for any b we have : $b \geq -a_-$.

Consider again the residual segment $(\underline{n'_r})$, and observe that by definition the sequence $(a, \dots, -a_-)$ is the longest linear segment with greatest (half)-integer a that one can withdraw from $(\underline{n'_r})$ such that the remaining segment $(\underline{n_{-a_-}})$ is a residual segment of the same type and $(a, \dots, -a_-)(\underline{n_{-a_-}})$ is in the Weyl group orbit of $(\underline{n'_r})$.

Further, this is true for any couple $(2a+1, 2a_- +1)$ of elements in the Jordan block associated to the residual segment $(\underline{n'_r})$.

It is therefore clear that given a and a_- such that $s_{-a_-} = \frac{a-a_-}{2} > 0$ is the smallest positive (half)-integers as possible, we have $s_b = \frac{a+b}{2} \geq s_{-a_-} = \frac{a-a_-}{2}$ and b is necessarily greater or equal to $-a_-$.

Once this embedding given, using Lemma 39 there exists an intertwining operator with non- generic kernel from this induced module to any other induced module from the cuspidal support $\bigotimes_{i\neq r} \sigma_i(\underline{n_i}) \otimes \rho(a, b, \underline{n_b})$ with $b \geq -a_-$.

By multiplicity one, it will also embed as a subrepresentation in the standard module $I_P^G(St_{t_b}(\rho)|.|^{s_b} \otimes \pi)$, with $s_b = \frac{a+b}{2}$ and t_b the length of the segment (a, b).

By the above notice, if γ appears as a submodule in the standard module $I_{P_\flat}^G(St_{t_{a_-}}(\rho)|.|^{s_{-a_-}} \otimes \pi_{a_-})$ with Langlands parameter s_{a_-}, it also appears as a submodule in any standard module

$$I_{P_\natural}^G(St_{t_b}(\rho)|.|^{s_b} \otimes \pi_b)$$

with $s_b \geq s_{-a_-}$ for the order defined in Lemma 31 as soon as $St_{t_b}(\rho)|.|^{s_b} \otimes \pi_b$ has equivalent cuspidal support. The parabolic subgroups $P_\flat, P_\natural$ are maximal parabolic subgroups which contain P_1.

$\square$

Remark 11. In the Weyl group W_σ orbit of $\rho(\underline{n'_r}) \bigotimes_{i\neq r}(\sigma_i(\underline{n}_i)\otimes\sigma_c)$, we could consider all cuspidal strings $(a, b, \underline{n})$ such that the induced modules $I_{P_1}^G(\rho(a, b, \underline{n}); (\sigma_i(\underline{n}_i))_{i\neq r}, \sigma_c)$ contain generic standard modules and such that $(\underline{n})$ is a dominant residual point of the same type as $(\underline{n'_r})$. Then any such generic standard module $I_P^G(St_n|.|^s \otimes \pi)$ should have γ has a subrepresentation.

More precisely, considering all couples (a,a_-) satisfying the conditions of Corollary 2, and therefore using point (2) of Corollary 2, we can obtain all cuspidal representations of M_1 in the cuspidal support

$$\sigma_\lambda := \rho(\underline{n'_r}) \bigotimes_{i\neq r}(\sigma_i(\underline{n}_i)) \otimes \sigma_c$$

They take the form

$$\rho(a, b) \otimes \sigma_r(\underline{n_r}) \bigotimes_{i\neq r}(\sigma_i(\underline{n}_i)) \otimes \sigma_c$$

with $a > b$, and therefore γ will embed in

$$I_{P_1}^G(\rho(a, b) \otimes \sigma_r(\underline{n_r}) \bigotimes_{i\neq r}(\sigma_i(\underline{n}_i)) \otimes \sigma_c)$$

with $n = a - b + 1$.

We can further extend this result on discrete series generic subquotient to **any** standard module ; i.e to any standard module induced from a **non necessarily maximal** parabolic subgroup.

Theorem 83. *Let us fix an irreducible cuspidal representation of a linear group ρ. Let*

$$\tau_s := Z(\rho, a_1, b_1) \otimes Z(\rho, a_2, b_2) \otimes \dots Z(\rho, a_t, b_t) \otimes \pi$$

be a generic irreducible essentially square- integrable representation of a standard Levi factor M of a classical group.

Segments are of length $n_i = a_i - b_i + 1$, parameters are $s_i = \frac{a_i+b_i}{2}$, and define

$$s = (s_1, s_2, \dots, s_t)$$

It is a stricly decreasing sequence of (half)-integers.

Then the cuspidal support of τ_s is given by

$$\sigma_\lambda := \rho(a_1, \dots, b_1)(a_2, \dots, b_2) \dots (a_t, \dots, b_t) \otimes \bigotimes_i \sigma_i(\underline{n_i}) \otimes \sigma_c$$

Let us assume the induced representation

$$I_{P_1}^G(\rho(a_1, b_1)(a_2, b_2) \dots (a_t, b_t); \sigma_i(\underline{n_i}); \sigma_c)$$

has a discrete series subquotient. Then, the unique irreducible generic discrete series subquotient of the standard module $I_P^G(\tau_s)$ where $P = MU$ is a subrepresentation.

Proof. Let us denote $P = GL \times P_c$. Let us write $\mathcal{S}(\rho, a_i, b_i) = \mathcal{S}_i$. Let us assume that for some indices $i, j \in \{1, \dots, t\}$, the segments $\mathcal{S}_i, \mathcal{S}_j$ are linked. Induction by parts yields :

$$I_{P_1}^G(\rho(a_1, b_1)(a_2, b_2) \dots (a_t, b_t); \sigma_i(\underline{n_i})) \cong I_P^G(I_{P_{1,GL}}^{GL}(\bigotimes_i \rho(a_i, b_i)) \otimes I_{P_{1,c}}^{M_c}(\bigotimes_i \sigma_i(\underline{n_i}) \otimes \sigma_c))$$

Then using Bernstein-Zelevinsky's Theorem 41, $I_{P_{1,GL}}^{GL}(\bigotimes_i \rho(a_i, b_i))$ is reducible.

The generalized injectivity conjecture for the linear group claims that π_{GL}, the unique generic irreducible subquotient appears as a subrepresentation. If $\pi_{GL} \hookrightarrow I_{P_{1,GL}}^{GL}(\bigotimes_i \rho(a_i, b_i))$, tensoring with π yields

$$\gamma \leq I_P^G(\pi_{GL} \otimes \pi) \hookrightarrow I_{P_1}^G(\rho(a_1, b_1)(a_2, b_2) \dots (a_t, b_t); \sigma_i(\underline{n_i}); \sigma_c)$$

The induced representation π_{GL} corresponds to the smallest Langlands parameter for the order defined in Lemma 31, as proven in Lemma 38.

We write

$$\pi_{GL} := I_{P'_{GL}}^{GL}(Z(\mathcal{S}'(\rho, a'_1, b'_1)) \otimes Z(\mathcal{S}'(\rho, a'_2, b'_2)) \otimes \dots \otimes Z(\mathcal{S}'(\rho, a'_t, b'_t))$$

where segments $\mathcal{S}'_i$'s are mutually unlinked, since the induced representation π_{GL} is irreducible by Zelevinsky's Theorem .

We are reduced to studying the case of any two segments $\mathcal{S}_i, \mathcal{S}_j$, j, i in $\{1, \dots, t\}$ unlinked.

When $\rho \not\cong \sigma_i$ for any i, since all segments $\mathcal{S}_i$, $i \in \{1, \dots, t\}$ are unlinked, we need $(a_i, b_i) = (a_j, b_j)$ for any $\{1, \dots, t\}$, and (a_i, b_i) is a residual point of a given type (see Proposition 77), the induced module $I_{P_1}^G(\sigma_\lambda) := I_{P_1}^G(\rho(a_1, b_1)(a_2, b_2) \dots (a_t, b_t); \sigma_i(\underline{n_i}); \sigma_c)$ has σ_λ dominant, and therefore already contains the unique generic subquotient as subrepresentation by Lemma 34.

Else $\rho \cong \sigma_j$ for one index j and there exists a Weyl group element $w \in W_\sigma$ such that :

$$w((a_1, \dots, b_1)(a_2, \dots, b_2) \dots (a_t, \dots, b_t)(\underline{n_j})) = (\underline{n'_j})$$

And then σ_λ is again a residual point by Proposition 78. We prove it by induction on the number of linear residual segments t appearing in the Levi subgroup.

Initialization : Assume $t = 0$, let $P^0 = G$, and π be the generic irreducible discrete series corresponding to the dominant residual point $\sigma_\lambda := \rho(\underline{n_r}) \otimes \bigotimes_i \sigma_{i \neq r}(\underline{n_i}) \otimes \sigma_c$.

We consider the module

$$I_{P^0}^G(\pi) \hookrightarrow I_{P_1}^G(\rho(\underline{n_r}) \otimes \bigotimes_{i \neq r} \sigma_i(\underline{n_i}) \otimes \sigma_c)$$

By Lemma 34, λ being in the closure of the positive Weyl chamber, the unique irreducible generic discrete series subquotient is necessarily a subrepresentation.

The proof of the step from $t = 0$ to $t = 1$ is Proposition 82.

Heredity :

Let $P_{n_1,n_2,\ldots,n_t}$ be any standard parabolic subgroup of G such that its Levi factor is a product of t linear groups and a smaller classical group of the same type as G.

Assume the result true for any standard module $I^G_{P_{n_1,n_2,\ldots,n_t}}(Z(\rho, a_1, b_1)\otimes Z(\rho, a_2, b_2)\otimes \ldots Z(\rho, a_t, b_t) \otimes \pi)$ with t or less than t linear residual segments.

Let γ be the irreducible generic discrete series subquotient of a standard module induced from a tensor product of $t+1$ linear essentially square-integrable representations, such that $s = (s_1, s_2, \ldots, s_t, s_{t+1})$, $s_{t+1} > 0$ is a stricly decreasing sequence of (half)-integers and any two segments $\mathcal{S}_i, \mathcal{S}_j$ with $i, j \in \{1, t+1\}$ are unlinked. Let π' be an irreducible generic discrete series of a classical group associated to the dominant residual point $\bigotimes_i \sigma_i(\underline{n_i}) \otimes \rho(\underline{n'_r}) \otimes \sigma_c$.

$$\gamma \leq I^G_{P_{n_1,n_2,\ldots,n_t,n_{t+1}}}(Z(\rho, a_1, b_1) \otimes Z(\rho, a_2, b_2) \otimes \ldots Z(\rho, a_{t+1}, b_{t+1}) \otimes \pi')$$

If the segment $\mathcal{S}_{t+1}$ can be inserted in $(\underline{n'_r})$ to obtain a residual point of the same type whose residual segment is $(\underline{n_r})$, then we set $\pi \hookrightarrow I^{M_c}_{P'_{1,c}}(\rho(\underline{n_r}) \otimes \bigotimes_{i\neq r} \sigma_i(\underline{n_i}))$ using Lemma 34 for an irreducible generic discrete series π of M_c (the classical part of the Levi $M_{n_1,n_2,\ldots,n_t}$).

Using the generalized injectivity for $t = 1$, this unique irreducible generic discrete series embeds in $I^{M_c}_{P'}(Z(\rho, a_{t+1}, b_{t+1}) \otimes \pi')$.

Using the induction hypothesis :

$$\gamma \hookrightarrow I^G_{P_{n_1,n_2,\ldots,n_t}}(Z(\rho, a_1, b_1) \otimes Z(\rho, a_2, b_2) \otimes \ldots Z(\rho, a_t, b_t) \otimes \pi)$$

and since $\pi \hookrightarrow I^{M_c}_{P'}(Z(\rho, a_{t+1}, b_{t+1}) \otimes \pi')$:

$$\gamma \hookrightarrow I^G_{P_{n_1,n_2,\ldots,n_t,n_{t+1}}}(Z(\rho, a_1, b_1) \otimes Z(\rho, a_2, b_2) \otimes \ldots Z(\rho, a_t, b_t) \otimes Z(\rho, a_{t+1}, b_{t+1}) \otimes \pi')$$

Else, assume the segment $\mathcal{S}_{t+1}$ cannot be inserted to form a residual segment $(\underline{n_r})$, using Lemma 59 we know there is at least one index $j \in \{1, \ldots, t\}$, such that $\mathcal{S}_j$ can be inserted into $(\underline{n'_r})$.

Then, by the exact same reasoning as developed in the previous paragraph, we have :

$$\gamma \hookrightarrow I^G_{P_{n_1,n_2,\dots,n_t,n_{t+1}}}(Z(\rho,a_1,b_1)\otimes Z(\rho,a_2,b_2)\otimes\dots\otimes Z(\rho,a_{t+1},b_{t+1})\otimes Z(\rho,a_j,b_j)\otimes\pi')$$
$$\hookrightarrow I^G_{P_1}(\bigotimes_{i\neq j}\rho(a_i,b_i)\otimes\rho(a_j,b_j)\otimes\bigotimes_i\sigma_i(\underline{n_i});\sigma_c) \quad (1.29)$$

Using induction in stages :

$$I^G_{P_1}(\bigotimes_{i\neq j}\rho(a_i,b_i)\otimes\rho(a_j,b_j)\otimes\bigotimes_i\sigma_i(\underline{n_i}))$$
$$\cong I^G_P(I^{GL}_{P_{1,GL}}(\bigotimes_{i\neq j}\rho(a_i,b_i)\otimes\rho(a_j,b_j))\otimes I^{M_c}_{P_{1,c}}(\bigotimes_i\sigma_i(\underline{n_i})\otimes\sigma_c))$$

Using Theorem 41, since any two segments $\mathcal{S}_i,\mathcal{S}_j$, j,i in $\{1,\dots,t\}$ are not linked,

$$I^{GL}_{P_{1,GL}}(\bigotimes_{i\neq j}\rho(a_i,b_i)\otimes\rho(a_j,b_j))$$

is irreducible and therefore isomorphic to $I^{GL}_{P_{1,GL}}(\otimes_{i=1}^{t+1}\rho(a_i,b_i))$.

Therefore

$$I^G_P(I^{GL}_{P_{1,GL}}(\bigotimes_{i\neq j}\rho(a_i,b_i)\otimes\rho(a_j,b_j))\otimes I^{M_c}_{P_{1,c}}(\bigotimes_i\sigma_i(\underline{n_i}))\otimes\sigma_c)$$
$$\cong I^G_{P_1}(\bigotimes_i\rho(a_i,b_i)\otimes\bigotimes_i\sigma_i(\underline{n_i})\otimes\sigma_c) \quad (1.30)$$

and we can eventually write :

$$\gamma \hookrightarrow I^G_{P_{n_1,n_2,\dots,n_t,n_{t+1}}}(Z(\rho,a_1,b_1)\otimes Z(\rho,a_2,b_2)\otimes\dots Z(\rho,a_t,b_t)\otimes Z(\rho,a_{t+1},b_{t+1})\otimes\pi')$$
$$\hookrightarrow I^G_{P_1}(\bigotimes_i\rho(a_i,b_i)\otimes\bigotimes_i\sigma_i(\underline{n_i})\otimes\sigma_c) \quad (1.31)$$

$\square$

Theorem 84. *Let π be an irreducible discrete series representation of a classical group M_c.*

Let ρ_i, $i\in\{1,\dots,t\}$, be irreducible cuspidal representations of linear groups and $\tau_s := Z(\rho_1,a_1,b_1)\otimes Z(\rho_2,a_2,b_2)\otimes\dots Z(\rho_t,a_t,b_t)\otimes\pi$ be an irreducible generic essentially square integrable representation of a Levi factor $M = M_LM_c$ of a classical group G. Let σ be an irreducible generic cuspidal representation of M_1 in the cuspidal support of τ.

Let us assume σ_λ is a residual point, then the unique irreducible generic discrete series subquotient of the standard module $I^G_P(\tau_s)\hookrightarrow I^G_{P_1}(\sigma_\lambda)$ is a subrepresentation.

Proof. Let $M_{1,c} = GL_{k_1} \times \ldots GL_{k_1} \times GL_{k_2} \times \ldots \times GL_{k_2} \times \ldots \times GL_{k_r} \times \ldots \times GL_{k_r} \times G(k)$ where $G(k)$ is a semi-simple group of absolute rank k of the same type as G. Let us denote $\sigma^c = \sigma_1 \otimes \ldots \sigma_1 \otimes \sigma_2 \otimes \ldots \sigma_2 \otimes \ldots \otimes \sigma_r \otimes \ldots \sigma_r \otimes \sigma_c$ an irreducible cuspidal generic representation in the cuspidal support of π, an irreducible discrete series representation of a classical group M_c.

Let d_i denote the number of factors equal to σ_i and let $(s_{i,j})_{i,j}$ be a family of non-negative real numbers, $1 \leq i \leq k$ and $1 \leq j \leq d_i$ for i fixed. Then, $\sigma_1|\cdot|^{s_{1,1}} \otimes \ldots \sigma_1|\cdot|^{s_{1,d_1}} \otimes \sigma_2|\cdot|^{s_{2,1}} \otimes \ldots \sigma_2|\cdot|^{s_{2,d_2}} \otimes \sigma_r|\cdot|^{s_{r,1}} \otimes \ldots \sigma_r|\cdot|^{s_{r,d_r}} \otimes \sigma_c$ is in the cuspidal support of π, where $\sigma_i \simeq \sigma_i^\vee$ for every i.

Therefore, the cuspidal support of τ_s is given by

$$\sigma_\lambda := \bigotimes_i \rho_i(a_i, \ldots, b_i) \bigotimes_j \sigma_j(\underline{n_j}) \otimes \sigma_c$$

Let us be more precise, we embed π in $I_{P_{1,c}}^{M_c}(\sigma_\nu^c)$ using the result of Heiermann-Opdam (Proposition 12).

$$\sigma_\nu^c := \underbrace{\sigma_1|.|^{\ell_1} \ldots \sigma_1|.|^{\ell_1}}_{n_{\ell_1,1} \text{ times}} \ldots \underbrace{\sigma_1|.|^0 \ldots \otimes \sigma_1|.|^0}_{n_{0,1} \text{ times}} \underbrace{\sigma_2|.|^{\ell_2} \ldots \sigma_2|.|^{\ell_2}}_{n_{\ell_2,2} \text{ times}} \ldots \underbrace{\sigma_2|.|^0 \ldots \otimes \sigma_2|.|^0}_{n_{0,2} \text{ times}}$$
$$\ldots \otimes \underbrace{\sigma_r|.|^{\ell_r} \ldots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r,r} \text{ times}} \ldots \underbrace{\otimes \sigma_r|.|^0 \ldots \otimes \sigma_r|.|^0}_{n_{0,r} \text{ times}} \otimes \sigma_c$$

Let us write $\lambda := (\lambda_1, \ldots, \lambda_{r+t})$, where the reducibility points s_j of $I_{P_1}^{G(k+k_i)}(\sigma_j|.|^{s_j} \otimes \sigma_c)$ (resp. $I_{P_1}^{G(k+k_i)}(\rho_j|.|^{s_j} \otimes \sigma_c)$) determines the type (i.e $B_{d_j}, C_{d_j}, D_{d_j}$) of the residual point λ_j, we have therefore two options for each λ_i :

$$\begin{cases} \textit{either } \rho_i \ncong \sigma_j \text{ for any } j, \text{ and } \lambda_i \text{ is a residual point if } (a_i, \ldots, b_i) \\ \text{satisfies the conditions of Proposition 77,} \\ \textit{or } \rho_i \cong \sigma_j \text{ for some } j \text{ and } \exists w \in W_\sigma \text{ such that } w((a_i, \ldots, b_i) + (\underline{n_i})) = \\ (\underline{n_i'}) \end{cases} \tag{1.32}$$

Using the result of Proposition 71, claiming that elementary intertwining operators interchanging cuspidal representations in two disjoint inertial orbits are one-to-one, we can reorganize the cuspidal representation

$$\sigma_\lambda := \bigotimes_i \rho_i(a_i, b_i) \bigotimes_j \sigma_j(\underline{n_j}) \otimes \sigma_c$$

as

$$\sigma_\lambda := \bigotimes_i \rho_i((a_i, b_i) + (\underline{n_i})) \bigotimes_k \sigma_k(\underline{n_k}) \otimes \sigma_c$$

where we have relabeled each segment $(\underline{n_j})$ such that $\sigma_j \cong \rho_i : (\underline{n_i})$.

Let γ be the irreducible generic subquotient of $I_P^G(\tau_s) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

It embeds as a subrepresentation in $I_{P_1}^G(\sigma'_{\lambda'})$, where $\sigma'_{\lambda'} := \bigotimes_i \rho_i((a'_i, \ldots, b'_i) + (\underline{n'_i})) \bigotimes_k \sigma_k(\underline{n_k}) \otimes \sigma_c$ where $(a'_i, \ldots, b'_i)$ remains $(a_i, \ldots, b_i)$ or is possibly empty as just explained in the Equation 1.32.

We consider intertwining operators between $I_{P_1}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$.

It is enough to understand the argument to go from

$$I_{P_1}^G(\rho_j((a'_j, b'_j) + (\underline{n'_j})) \bigotimes_{i \neq j} \sigma_i(a'_i, b'_i, \underline{n'_i}) \bigotimes_k \sigma_k(\underline{n_k}) \otimes \sigma_c)$$

to $I_{P_1}^G(\rho_j((a_j, b_j) + (\underline{n_j})) \bigotimes_{i \neq j} \sigma_i(a'_i, b'_i, \underline{n'_i}) \bigotimes_k \sigma_k(\underline{n_k}) \otimes \sigma_c)$ and repeat the same arguments for all j.

Possibly, for certain indices i the above intertwining operator has non-generic kernel. For all others indices i, we argue as follows :

$$\gamma \hookrightarrow I_{P'}^G(Z(\rho_1, a'_1, b'_1) \otimes Z(\rho_2, a'_2, b'_2) \otimes \ldots Z(\rho_t, a'_t, b'_t) \otimes \pi') \hookrightarrow I_{P_1}^G(\sigma'_{\lambda'}) \quad (1.33)$$

where π' is an irreducible generic discrete series embedded in $I_{P_{1,c}}^{M'_c}(\bigotimes_i \sigma_i(\underline{n'_i}))$.

Choose one index j as above, and using the generalized injectivity conjecture as proved in Proposition 82, we can embed π' in $I_{P_{1,c}}^{M'_c}(\rho_j((a^\sharp, b^\sharp) + (\underline{n''_j})) \bigotimes_{i \neq j} \sigma_i(\underline{n'_i}) \otimes \sigma_c)$.

Therefore

$$\begin{aligned}\gamma \hookrightarrow I_{P_1}^G(Z(\rho_1, a'_1, b'_1) \otimes Z(\rho_2, a'_2, b'_2) \otimes \ldots Z(\rho_t, a'_t, b'_t) \otimes \pi') \\ \hookrightarrow I_{P_1}^G(\otimes_{i \neq j} \rho_i(a'_i, b'_i) \otimes I_{P_{1,c}}^{M'_c}(\rho_j((a^\sharp, b^\sharp) + (\underline{n''_j})) \bigotimes_{i \neq j} \sigma_i(\underline{n'_i})) \otimes \sigma_c) \\ \cong I_{P_1}^G(\otimes_{i \neq j} \rho_i(a'_i, b'_i) \otimes \rho_j((a^\sharp, b^\sharp) + (\underline{n''_j})) \bigotimes_{i \neq j} \sigma_i(\underline{n'_i}) \otimes \sigma_c) \quad (1.34)\end{aligned}$$

The choice of $(a^\sharp, b^\sharp)$ is determined to insure we have an intertwining operator with non-generic kernel from $I_{P_1}^G(\otimes_{i \neq j} \rho_i(a'_i, b'_i) \otimes \rho_j((a^\sharp, b^\sharp) + (\underline{n''_j})) \bigotimes_{i \neq j} \sigma_i(\underline{n'_i}) \otimes \sigma_c)$ to

$$I_{P_1}^G(\otimes_{i \neq j} \rho_i(a'_i, b'_i) \otimes \rho_j((a_j, b_j) + (\underline{n_j})) \bigotimes_{i \neq j} \sigma_i(\underline{n'_i}) \otimes \sigma_c)$$

□

D.4.3. Proof of the Generalized Injectivity Conjecture for Non-Discrete Series Subquotients

Proposition 85. *Let $\tau_{s\tilde{\alpha}} := St_n(\rho)|.|^s \otimes \pi$ be an irreducible generic essentially square-integrable representation of a maximal standard Levi subgroup M of a* *classical group G.*

Denote $\sigma_\lambda := \rho(a, \dots, b) \otimes \bigotimes_i(\sigma_i(\underline{n}_i)) \otimes \sigma_c$ its cuspidal support, where $s = \frac{a+b}{2}$.

Assume λ is not a residual point, and therefore the unique irreducible generic subquotient in $I_P^G(St_n(\rho)|.|^s \otimes \pi)$ is tempered or non- tempered.

The unique tempered or non-tempered generic subquotient of the standard module $I_P^G(\tau_{s\tilde{\alpha}})$ is a submodule.

Proof. If $\rho \not\cong \sigma_i$ for all i, we have seen $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a residual point if and only if each λ_i is. Assume $\lambda_1 := (a, \dots, b)$ is not a residual point of type $C_{d_1}, B_{d_1}, D_{d_1}$, as given in Proposition 78. Then $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is not a residual point and therefore the unique irreducible generic subquotient in the standard module $I_P^G(St_n(\rho)|.|^s \times \pi) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$ is tempered or non-tempered.

Let us assume, we are the case of unramified principal series, $P_1 = P^0, \sigma = \mathbf{1}$, then we can reorganize the cuspidal support so as to find a minimal Langlands parameter $\nu' \leq s$ for the order defined in Definition 30 and set $I_{P'}^G(\tau'_{\nu'})$ to be this unique irreducible generic subquotient.

If $P_1 \neq P^0$, let us write the cuspidal support $\sigma_\lambda := \rho(a, b) \otimes \bigotimes_i(\sigma_i(\underline{n}_i)) \otimes \sigma_c$

Since $\rho \not\cong \sigma_i$ for all i, there is no way we can reorganize the cuspidal support ; which means this standard module has no irreducible generic subquotient, but itself.

Else, if $\rho \cong \sigma_j$ for one index j. Assume as before that $j = r$, but let us denote $(\underline{n})$ the residual segment $(\underline{n_r})$.

We assume $I_P^G(St_n|.|^s \otimes \pi) \hookrightarrow I_{P_1}^G(\rho(a, b, \underline{n}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$ has a unique irreducible generic subquotient.

By Langlands' classification and the Standard module conjecture (see the Subsection 1.2.4) there is a discrete series τ' and a Langlands parameter ν' such that this unique irreducible generic subquotient has the form $J_{P'}^G(\tau'_{\nu'}) \cong I_{P'}^G(\tau'_{\nu'})$.

By Theorem 35, ν' corresponds to the minimal Langlands parameter for a given cuspidal support, $\nu' < s\tilde{\alpha}$.

For an explicit description of the parameter ν', given the cuspidal string $(a, b, \underline{n})$, the reader is encouraged to read the analysis conducted in Appendix G.

Then, τ' (for e.g $St_q|.|^{\nu'} \otimes \pi'$, for a given integer q) corresponds to a cuspidal support $(a', b', \underline{n'})$, that is :

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P_1}^G(\rho(a', b', \underline{n'}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$$

It is enough to understand how one passes from the cuspidal string $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ to understand the strategy for embedding the unique irreducible generic subquotient as a subrepresentation in $I_P^G(St_n|.|^s \otimes \pi)$.

Starting from $(a, b, \underline{n})$, to minimize the Langlands parameter ν', we usually remove elements at the end of the first segment (i.e. the segment $(a, \dots, b)$ to insert them on the second residual segment, or we enlarge the first segment on the right. This means either $a' < a$, or $b' < b$, or both.

If $a' = a$, and $b' < b$, in particular if $b' < 0$, we have a non-generic kernel operator between $I_{P_1}^G((a', b', \underline{n'}), \sigma_c)$ and $I_{P_1}^G((a, b, \underline{n}), \sigma_c)$ as proved in Lemma 39.

Otherwise, one observes that passing from $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ require certain elements as γ in the above residual segment with $a \geq \gamma > a'$ to move up, i.e. from right to left. This means using rank one operators which change $(\gamma + n, \gamma)$ to $(\gamma, \gamma + n)$ for integers $n \geq 1$, those rank one operator may clearly have generic kernel.

In this context, we will rather use the results of Proposition 82, and Theorem 83.

Consider again $I_{P'}^G(\tau'_{\nu'})$ and the cuspidal support : $\rho(a', b', \underline{n'}) \otimes \bigotimes_{i \neq r} \sigma_i(\underline{n_i}) \otimes \sigma_c$

Consider the irreducible generic discrete series π' corresponding to the dominant residual point $(\underline{n'} + \bigoplus_{i \neq r}(\underline{n_i}))$, it is the unique irreducible generic subquotient embedded in the representation induced from $\rho(\underline{n'}) \otimes \bigotimes_{i \neq r} \sigma_i(\underline{n_i}) \otimes \sigma_c$.

$$\pi' \hookrightarrow I_{P_1}^{M'_c}(\rho(\underline{n'}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$$

By the generalized injectivity conjecture for discrete series subquotient (Theorem 83), any standard module embedded in $I_{P_1}^{M'_c}(\rho(\underline{n'}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$ has π' as subrepresentation.

We may therefore embed π' as subrepresentation in

$$I_{P_1}^{M'_c}(\rho((a^\flat, b^\flat) + (\underline{n}^\flat)) \bigotimes_{i \neq r} \sigma_i(\underline{n_i}) \otimes \sigma_c)$$

and therefore

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P_1}^G(\rho((a', b') + (a^\flat, b^\flat) + (\underline{n}^\flat) \bigotimes_{i \neq r} \sigma_i(\underline{n_i}) \otimes \sigma_c)$$

The sequence $(a^\flat, b^\flat) + (\underline{n}^\flat)$ is chosen appropriately to have a non-generic kernel operator from $I_{P_1}^G(\rho((a', b') + (a^\flat, b^\flat) + (\underline{n}^\flat)); \bigotimes_i \sigma_i(\underline{n_i}))$ to $I_{P_1}^G(\rho(a, b, \underline{n}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$.

The unique irreducible generic subrepresentation $I_{P'}^G(\tau'_{\nu'})$ in

$I_{P_1}^G(\rho(a, b, \underline{n}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$ cannot appear in the kernel and therefore appears in the image of this operator. It therefore appears as a subrepresentation in $I_{P_1}^G(\rho(a, b, \underline{n}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}), \sigma_c)$ and by multiplicity one of the generic piece in $I_{P_1}^G(\rho(a, b, \underline{n}); \bigotimes_{i \neq r} \sigma_i(\underline{n_i}))$, it also appears as subrepresentation in the standard module $I_P^G(St_n|.|^s \otimes \pi)$. $\square$

Theorem 86. *Let us fix an irreducible cuspidal representation of a linear group ρ. Let*

$$Z(\rho, a_1, b_1) \otimes Z(\rho, a_2, b_2) \otimes \dots Z(\rho, a_t, b_t) \otimes \pi$$

be an irreducible generic essentially square integrable representation of a standard Levi subgroup $M_{n_1,n_2,\ldots,n_t}$ of a classical group G.

Its cuspidal support is given by

$$\sigma_\lambda := \rho(a_1,\ldots,b_1)(a_2,\ldots,b_2)\ldots(a_t,\ldots,b_t)\bigotimes_i \sigma_i(\underline{n_i})$$

Assume λ is not a residual point, then the unique irreducible generic subquotient in

$$I^G_{P_{n_1,n_2,\ldots,n_t}}(Z(\rho,a_1,b_1)\otimes Z(\rho,a_2,b_2)\otimes\ldots Z(\rho,a_t,b_t)\otimes\pi)$$

is tempered or non-tempered. The unique irreducible generic tempered or non-tempered subquotient is a submodule.

Proof. Let $P_{n_1,n_2,\ldots,n_t}$ be any standard parabolic subgroup of G such that its Levi factor, $M_{n_1,n_2,\ldots,n_t}$, is a product of t linear groups and a smaller classical group of the same type as G.

The proof goes along the same inductive line than in the proof of Theorem 83.

The case of $t=1$, $I^G_P(St_n|.|^s\otimes\pi)$ is Proposition 85. That is, given a cuspidal support (P_1,σ_λ), for any standard module induced from a maximal parabolic subgroup $P: I^G_P(St_n|.|^s\otimes\pi)\hookrightarrow I^G_{P_1}(\sigma_\lambda)$, the unique irreducible generic subquotient is a subrepresentation.

Considering that a tempered or non-tempered irreducible generic subquotient in a standard module with t linear residual segments

$$I^G_{P_{n_1,n_2,\ldots,n_t}}(Z(\rho,a_1,b_1)\otimes Z(\rho,a_2,b_2)\otimes\ldots Z(\rho,a_t,b_t)\otimes\pi)$$

is necessarily a subrepresentation; one uses the same arguments than in the proof of Theorems 83 and 84 to conclude that a tempered or non- tempered irreducible generic subquotient in a standard module with $t+1$ linear residual segments

$$I^G_{P_{n_1,n_2,\ldots,n_t,n_{t+1}}}(Z(\rho,a_1,b_1)\otimes Z(\rho,a_2,b_2)\otimes\ldots Z(\rho,a_t,b_t)\otimes Z(\rho,a_{t+1},b_{t+1})\otimes\pi)$$

is a subrepresentation, therefore proving the theorem. □

D.5. Examples

D.5.1. Generic discrete series subquotient

Example 16. Let $Z(\rho,a,b)\times\tau$ be a generic discrete series of $G=G(n)$, for ρ a cuspidal representation of a linear group, and τ an irreducible generic discrete series of $G(n_0)$.

By Heiermann-Opdam's Result [Proposition 12] and Lemma 34 and the explanations of section 1.5, τ is the generic discrete series uniquely corresponding to a dominant residual point $(\sigma_i(\underline{n_i}))_i$, more precisely, there exists a parabolic subgroup

P_1^c such that $\tau \hookrightarrow I_{P_1^c}^{G(n-n_1)}((\sigma_i(\underline{n}_i))_i)$ The form of the residual segments $(\underline{n}_i)$ does not interest us (we know their type (i.e $B_{d_i}, C_{d_i}, D_{d_i}$) depend on the reducibility point (0,1/2, or 1) of the induced representation of $G(n_0+n_i)$, $I_{P_1}^{G(n_0+n_i)}(\sigma_i|.|^s \otimes \sigma_c)$), except for the unique representation isomorphic to the cuspidal representation ρ, assumed without loss of generality to be σ_r.

Let the residual segment $(\underline{n}_r)$ be (7654322110), it is a residual segment of type B_{10}. Its Jordan block' s elements are $\{15, 5, 3\}$. Suppose first that $Z(\rho, a, b) \hookrightarrow I_{P_1^L}^{GL_{n_r}}(\rho, (6543210-1-2-3-4))$. That is :

$$Z(\rho, 6, -4) \times \tau \hookrightarrow I_{P_1}^G(\rho(6, \ldots, -4); (\sigma_i(\underline{n}_i))_i)$$

In the Weyl group orbit of the sequence : (6543210-1-2-3-4)(7654322110), the dominant residual point of type B_{21} is : (766554443332222111100) ; its Jordan block' s elements are $\{15, 9, 7, 5, 3\}$. By Heiermann-Opdam's Result 12 and Lemma 34, the induced module

$$I_{P_1}^G(\rho(766554443332222111100); (\sigma_i(\underline{n}_i))_i)$$

has a unique irreducible generic subquotient which is discrete series and embeds as a subrepresentation :

$$\gamma \hookrightarrow I_{P_1}^G(\rho(766554443332222111100); (\sigma_i(\underline{n}_i))_i)$$

since

$$(766554443332222111100)(\underline{n}_i))_i$$

is a dominant residual point.

The content of Proposition 56 and Corollary 2, point (2) claim that for any two consecutive elements in the Jordan block of γ : $\{2a_i + 1, 2a_{i+1} + 1\}$, γ should embed as a subrepresentation in $Z(\rho, a_i, a_{i+1} + 1) \times \theta_i$ for an irreducible representation θ_i and further by Moeglin's Lemma 79 in $Z(\rho, a_i, -a_{i+1}) \times \tau_i$ for a generic discrete series τ_i.

$$\gamma \hookrightarrow Z(\rho, a_i, -a_{i+1}) \times \tau_i \hookrightarrow I_{P_1}((a_i, \ldots, -a_{i+1})(\underline{n}'))$$

We give the argumentation for the two consecutive elements $\{9, 7\}$ (for all others consecutive elements in the Jordan block the same reasoning applies).

It is also harmless to do the reasoning only on the cuspidal string (766554443332222111100) and forget about all other residual segments constituing the cuspidal support $(\underline{n}_i))_{i\neq r}$. Further, as in the proof of Proposition 56, since we fix ρ, it is omitted in the following argumentation.

$$\gamma \hookrightarrow \nu^7 \times \nu^6 \times \nu^6 \times \nu^5 \times \nu^5 \times \nu^4 \times \nu^4 \times \nu^4 \times \nu^3 \times \nu^3 \times \nu^3 \times \nu^2 \times \nu^2 \times \nu^2 \times \nu^2 \times \nu^1 \times \nu^1 \times \nu^1 \times \nu^1 \times \nu^0 \times \nu^0 \rtimes \sigma_c$$

Denote Π_1 this right hand representation.

$$\Xi_1 = Z[6,7]\times\nu^6\times\nu^5\times\nu^5\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\nu^2\times\nu^2\times\nu^2\times\nu^1\times\nu^1\times\nu^1\times\nu^1\times\nu^0\times\nu^0\rtimes\sigma_c$$

is a subrepresentation in Π_1.

Using GL theory, the representation Ξ_1 is generic and therefore Ξ_1 has a unique irreducible generic subquotient by Rodier's theorem. If γ does not embed in Ξ_1 then it embeds in Π_1/Ξ_1. Further, the unique irreducible generic subquotient of Ξ_1 we denote π'. But then Π_1 would contain γ and γ' contradicting Rodier's theorem. Therefore

$$\gamma \hookrightarrow \Xi_1 \cong \nu^6\times Z[6,7]\times\nu^5\times\nu^5\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\ldots\times\nu^1\times\ldots\times\nu^0\times\nu^0\rtimes\sigma_c = \Pi_2$$

$$\Xi_2 = Z[5,7]\times\nu^6\times\nu^5\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\nu^2\times\nu^2\times\nu^2\times\nu^1\times\nu^1\times\nu^1\times\nu^1\times\nu^0\times\nu^0\rtimes\sigma_c$$

embeds as a subrepresentation in Π_2 and again using the fact that Ξ_2 is generic and Rodier's Theorem,

$$\gamma \hookrightarrow \Xi_2 = Z[5,7]\times\nu^6\times\nu^5\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\nu^2\times\ldots\rtimes\sigma_c \tag{1.35}$$
$$\cong \nu^6\times\nu^5\times Z[5,7]\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\nu^2\times\ldots\rtimes\sigma_c \tag{1.36}$$

Since (6,5) and (7,6,5) are linked segments.

The same argumentation gives finally : $\gamma \hookrightarrow \Xi_3 = Z[5,6]\times Z[5,7]\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\nu^2\times\nu^2\times\nu^2\times\nu^1\times\nu^1\times\nu^1\times\nu^1\times\nu^0\times\nu^0\rtimes\sigma_c := Z[5,6]\times\Theta_3$

One can further consider a generic filtration of $\Theta_3 = Z[5,7]\times\nu^4\times\nu^4\times\nu^4\times\nu^3\times\nu^3\times\nu^3\times\nu^2\times\nu^2\times\nu^2\times\nu^2\times\nu^1\times\nu^1\times\nu^1\times\nu^1\times\nu^0\times\nu^0\rtimes\sigma_c$: $V_0 = \{0\}\subseteq V_1\subseteq V_2\subseteq\ldots$ where V_{i+1}/V_i is irreducible for any i.

Assume

$$\gamma \not\hookrightarrow Z[5,6]\times V_1 \tag{1.37}$$

then $\gamma\hookrightarrow Z[5,6]\times\Theta_3/V_1$ and $V_1/V_2\subseteq\Theta_3/V_1$

If

$$\gamma \not\hookrightarrow Z[5,6]\times V_2/V_1 \tag{1.38}$$

then from (1.37) and (1.38)

$$\gamma \not\hookrightarrow Z[5,6]\times V_2$$

Therefore $\gamma\hookrightarrow Z[5,6]\times\Theta/V_2$; repeating this procedure we eventually need to have some index j so that

$$\gamma \hookrightarrow Z[5,6]\times V_{j+1}/V_j$$

And eventually, using Moeglin's Lemma, $\gamma\hookrightarrow Z[-4,6]\times\tau_6$ for a generic discrete series τ_6.

D.5.2. Generic non-discrete series subquotient

Example 17. We continue with the setting of example (16) but change a and b to (2) and (-1). Then, $Z(\rho, a, b) \hookrightarrow I_{P_1^L}^{GL_{nr}}(\rho, (210-1))$, so that :

$$Z(\rho, 2, -1) \times \tau \hookrightarrow I_{P_1}^G(\rho(210-1); (\sigma_i(\underline{n}_i))_i)$$

(210-1) cannot be inserted in (7654322110) to form a residual point (indeed 5 and 3 are already in the Jordan block of the latter). Since (210-1)(7654322110) is not a residual point, the unique irreducible generic subquotient is not discrete series ; and considering our study of minimal Langlands parameter for the order given in Lemma 31 corresponding to the unique generic subquotient, this generic subquotient is :

$$I_{P'}^G(\tau_{\nu'}) := I_{P'}^G(\tau'_{\nu=0}) \hookrightarrow I_{P_1}^G(\rho((210-1-2)(765432110)); \sigma_i(\underline{n}_i))_i)$$

We consider further the intertwining operator between

$$I_{P_1}^G(\rho((210-1-2)(765432110)); \sigma_i(\underline{n}_i))_i)$$

and $I_{P_1}^G(\rho((210-1)(7654322110)); \sigma_i(\underline{n}_i))_i)$.

Since $(-2) > i$ for any $i \in \{7, 0\}$, it is composed of rank one operators with non-generic kernel and therefore it has non-generic kernel. The unique generic subquotient $I_{P'}^G(\tau_{\nu'})$ appears in the image of this operator, and therefore as a subrepresentation in $I_{P_1}^G(\rho((210-1)(7654322110)); \sigma_i(\underline{n}_i))_i)$ and therefore (by multiplicity one of the irreducible generic piece in this induced module) in $Z(\rho, 2, -1) \times \tau$.

Finally consider this example to illustrate the second part of the proof of Proposition 61.

Example 18. Consider the standard module

$$Z(\rho, 5, -2) \times \tau \hookrightarrow I_{P_1}^G((\rho(543210-1-2)(43221110); (\sigma_i(\underline{n}_i))_i))$$

The irreducible generic subquotient is $I_{P'}^G(Z(\rho, 2, -2) \times \tau') \hookrightarrow I_{P_1}^G(\rho(210-1-2)(54433221110))$ We need to study the operator going from the cuspidal support (210-1-2)(54 43 32 211 1 0) to (543210-1-2)(4 32 211 1 0). It is easy to observe that bringing 'up' the (5,4) uses non-generic kernel operator, however to bring 'up' the '3', one would need rank one operators $(4, 3) \to (3, 4)$ and $(5, 3) \to (3, 5)$ which may have generic kernel.

We therefore embed τ' in $I_{P_1'}^{M'}(\rho(543)(43221110))$ as a subrepresentation. Then,

$$I_{P'}^G(Z(\rho, 2, -2) \times \tau') \hookrightarrow I_{P_1''}^G(\rho(210-1-2)(543)(43221110))$$

And one easily checks that there is a non-generic kernel operator from

$$I^G_{P_1''}(\rho(210-1-2)(543)(43221110)) \qquad \text{to} \qquad I^G_{P_1}(\rho(543210-1-2)(43221110))$$

E. Projections of roots systems

Let us first follow the notations of the book of Renard 2010, Chapter V. We will also use the notations of the Section 1.2. Let $X^*(G)$ denote the group of rational characters of G; it dual is $X_*(G)$. Let A_M be the split component in M and A_0 the maximal split component in M_0. We denote $a_0 = X_*(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $a_0^* = X^*(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$.

The duality between $X^*(A_0)$ and $X_*(A_0)$ extends to a duality (canonical pairing) between the vector spaces a_0 and a_0^*. We have the following diagram (see the Chapter V of Renard 2010) :

$$\begin{tikzcd} a_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r] & X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} = a_M^* \arrow[d] \\ a_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r] \arrow[u] & X^*(A_G) \otimes_{\mathbb{Z}} \mathbb{R} = a_G^* \end{tikzcd}$$

The horizontal arrows are isomorphisms. If we denote a_M^G the kernel of the vertical arrow on the right, we obtain :

$$a_M^* = a_G^* \oplus (a_M^G)^*$$

And in the dual :

$$a_M = a_G \oplus (a_M^G)$$

Let M be a standard Levi subgroup of G such that the set of simple roots in Lie(M) is $\Delta_M = \Theta$. Let us therefore denote $a_M = a_\Theta$.

Because of the existence of the scalar product (sustaining the duality), the restriction map from $(a_0^G)^*$ to $(a_\Theta^G)^*$ is a *projection* map from (a_0^G) to (a_Θ^G). With the notations of the Subsection 1.8.1, the roots in $\Delta(P_1)$ generating $(a_{M_1})^*$ are non -trivial restrictions of roots in $\Delta \setminus \Delta^{M_1}$ [7], and (a_{M_1}) is generated by the projection of roots in $\Delta^\vee \setminus \Delta^{M_1 \vee}$.

In this Appendix, we will rather consider projections of roots.

Let a be a real euclidian vector space of finite dimension and Σ a root system in a with a basis Δ. Let $\Theta \subset \Delta$, to avoid trivial cases we assume Θ is a proper subset of Δ, i.e. that Θ is neither empty nor equal to Σ. Let us consider the projection of Σ on a_Θ and we denote Σ_Θ the set of all non-trivial projections of roots in Σ. Our context is that of $a = a_0^G := a_0/a_G$ quotient of the Lie algebra of the maximal split torus A_0 by the Lie algebra of the center of G. We consider Σ as root system of G, an order, and a basis Δ. Let M be a standard Levi subgroup

7. Recall that in the notations of Walspurger.JL 1995, I.1.6, Δ^{M_1} are the roots of Δ which are in M_1

of G such that the set of simple roots in Lie(M) is $\Delta^M = \Theta$. Then $a_\Theta = a_M/a_G$. We don't consider the trivial case where $M = M_0$ and $M = G$. Let us denote d the dimension of a_Θ, i.e the cardinal of $\Delta - \Theta$.

Let us also denote Δ_Θ the set of projections of the simple roots in $\Delta - \Theta$ on a_Θ. In general Σ_Θ is not a root system, however let us observe :

Lemma 87. *The elements in Σ_Θ are, in a unique way, linear combination with entire coefficients all with the same sign of the elements in Δ_Θ.*

We would like to determine the conditions under which Σ_Θ contains a root system (for a subspace of a_Θ) and what are the types of root system appearing. We will classify the subsystems of rank d appearing when they exist. Of course, there are always subsystems of rank 1 and as Θ is assumed to be non-empty there is no need to discuss the case where Σ is of rank 2 (in particular G_2). We will therefore consider the root systems Σ of rank $n \geq 3$ and $d \leq n-2$. Let us remark that we will find irreducible non reduced root systems : they are the BC_d which contain three subsystems of rank d : B_d, C_d and D_d.

We will use the following remark (see the Chapter VI in *Groupes et Algèbres de Lie, Chapitre 4,5, et 6*, in particular Equation (10) in VI.3 and Proposition 12 in VI.4). Let α and β be two non-orthogonal elements of a root system. Set

$$C = \left(\frac{1}{\cos(\alpha,\beta)}\right)^2 \qquad \text{and} \qquad R = \frac{||\alpha||^2}{||\beta||^2} .$$

The only possible values for C (the inverse of the square of the cosinus of the angle between two roots) are 4, 2 and $\frac{4}{3}$ whereas assuming the length of α larger or equal to the one of β, the quotient of the length is respectively 1, 2 or 3. Thus, if $||\alpha|| \geq ||\beta||$

$$\frac{C}{R} \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = 4 .$$

We will therefore compute the quotient of length and the angles of the non-trivial projections of roots in Σ, in particular those of elements in $\Delta - \Theta$.

The main result

Theorem 88. *Let Σ be an irreducible root system of classical type (i.e of type A, B, C or D). The subsystems in Σ_Θ are necessarily of classical type. In addition, if the irreducible (connected) components of Θ of type A are all of the same length, the interval between each of them of length one, then Σ_Θ contains an irreducible root system of rank d (non necessarily reduced).*

We will prove this theorem *via* a case-by-case analysis.

E.1. The case of Σ of type A

Let us consider a_0 to be of dimension $n+1$ and with orthonormal basis $e_1, e_2, \ldots, e_{n+1}$. Let us denote Ξ this ordered basis, i.e the ordered set of the e_i. The elements of Σ are the $e_i - e_j$ with $i \neq j$; they generate a subspace a of dimension n and Δ is the set of simple roots $\alpha_i = e_i - e_{i+1}$. Let us denote $\overline{e_i}$ the projection of e_i on a_Θ. The Dynkin diagram of Θ is a union of irreducible (or connected) components of type A. Therefore, the data of Θ corresponds to a partition of the ordered set Ξ in a disjoint (ordered) union of ordered parts that we index by the smallest index appearing in the indices of the basis vectors associated :

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_l \ .$$

The correspondence is defined as follows, the part :

$$\Xi_r = \{e_r, \cdots, e_{r+m}\}$$

is associated to the empty subset if $m = 0$ and to the subset of simple roots

$$\{\alpha_r, \cdots, \alpha_{r+m-1}\} \qquad \text{si } m \geq 1 \ .$$

Let us consider an element e_i in the basis Ξ of a_0. Let r be the smallest integer j such that $\overline{e_j} = \overline{e_r}$, and let $r+m$ be the largest. We will have $\overline{e_k} = \overline{e_i}$ for any k such that $r \leq k \leq r+m$. If $m = 0$, it is clear. Observe that if $m = 0$, the two simple consecutive roots α_{i-1} and α_i where e_i appears are outside Θ. Now, let $m \geq 0$, the root $e_r - e_{r+m}$ has a trivial projection on a_Θ and therefore by Lemma 87 all the simple roots that occur in the expression of this root shall be in Θ. As a result, the roots $\alpha_k = e_k - e_{k+1}$ belong to Θ for any k such that $r \leq k \leq r+m-1$ and we have :

$$\overline{e_k} = \frac{e_r + e_{r+1} + \cdots + e_{r+m}}{m+1}$$

for all k with $r \leq k \leq r+m$.

Indeed, this expression of $\overline{e_k}$ is then orthogonal to all the roots $\alpha_k = e_k - e_{k+1}$ for any k such that $r \leq k \leq r+m-1$.

Such a chain of simple roots is a connected component of length m of the Dynkin diagram associated to Θ. We have observed that such a connected component is empty when e_r is orthogonal to all the elements in Θ in which case $m = 0$ i.e the two consecutive simple roots α_{r-1} and α_r are outside Θ. If e_r is associated to a length m connected component of Θ and therefore belongs to an ordered part of cardinal $m+1$ of Ξ, the square of the length of $\overline{e_r}$ is :

$$||\overline{e_r}||^2 = \frac{1}{m+1} \ .$$

Let us consider three vectors e_r, e_s and e_t whose projections $\overline{e_r}, \overline{e_s}$ and $\overline{e_t}$ are distinct and are associated to three components of Θ of type A_m, A_p and A_q. Let $\alpha = e_i - e_j$ a root whose projection

$$\overline{\alpha} = \pm(\overline{e_r} - \overline{e_s}) \ .$$

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \ .$$

Let us consider a root $\beta = e_k - e_l$ whose projection is

$$\overline{\beta} = \pm(\overline{e_s} - \overline{e_t})$$

we will obtain

$$||\overline{\beta}||^2 = \frac{1}{p+1} + \frac{1}{q+1}$$

and the square of the scalar product of $\overline{\alpha}$ and $\overline{\beta}$ is

$$\left(< \overline{\alpha}, \overline{\beta} >\right)^2 = \frac{1}{(p+1)^2} \ .$$

Thus we have :

$$C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\beta})}\right)^2 = \left(1 + \frac{p+1}{m+1}\right)\left(1 + \frac{p+1}{q+1}\right)$$

and if we assume $||\overline{\beta}|| \geq ||\overline{\alpha}||$ i.e $q \geq m$, we have :

$$R = \frac{||\overline{\alpha}||^2}{||\overline{\beta}||^2} = \frac{\left(1 + \frac{p+1}{m+1}\right)}{\left(1 + \frac{p+1}{q+1}\right)}$$

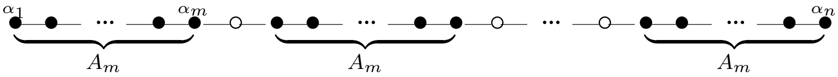
Then

$$\frac{C}{R} = \left(1 + \frac{p+1}{q+1}\right)^2 \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = \left(1 + \frac{p+1}{m+1}\right)^2 = 4 \ .$$

The only possible case is $C/R = 4$ and thus $R = 1$ and $C = 4$. This implies $m = p = q$ and $\{\overline{\alpha}, \overline{\beta}\}$ generate a root system of type A_2 : $\pm(\overline{e_r} - \overline{e_s})$, $\pm(\overline{e_s} - \overline{e_t})$ and $\pm(\overline{e_r} - \overline{e_t})$.

Lemma 89. *If Σ is of type A_n the only irreducible subsystems appearing in Σ_Θ are of type A. To have a root system of rank the dimension d of $\mathfrak{a}_\Theta$ it is necessary if $d > 1$, that the Dynkin diagram of Θ be a disjoint union of $d+1$ connected components of type A_m with $m \geq 0$, the intervals between each such component being of length one :*

$$n + 1 = (m+1)(d+1)$$



This corresponds to a partition of the ordered basis Ξ in an union of $d+1$ ordered parts of cardinal $m+1$:

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_{d+1}$$

where

$$\Xi_r = \left\{e_{(r-1)(m+1)+1} \cdots e_{r(m+1)}\right\} .$$

In this case Σ_Θ is of type A_d.

Proof. An irreducible subsystem is necessarily generated by the projections of roots of the form $\overline{\alpha} = \overline{e_i} - \overline{e_j}$ where the vecteurs $\overline{e_*}$ are all of the same length ; When we order these vectors following the $d+1$ indices, we obtain a basis of a subspace b_0 of a_0 containing a subspace b of codimension one in which the $\overline{e_i} - \overline{e_j}$ generate a system of type A. The rest of the corollary follows easily. □

E.2. The case B_n

In this case the basis of a is constituted of the e_i for $i \in \{1, \cdots, n\}$ and the elements in Σ are the $\pm e_i$ and the $\pm e_i \pm e_j$ and Δ is formed of the $\alpha_i = e_i - e_{i+1}$ for $i \leq n-1$ and of $\alpha_n = e_n$. The set Θ ia an union of irreducible components which are all of type A except for at most one which is of type B_r.

We distinguish two cases according to whether e_n belongs to Θ or not, i.e according to whether one of the components is of type B or not (case $r = 0$).

Case 1 ($r = 0$) : $e_n \notin \Theta$. In this case Θ is an union of components of type A. As in the case previously treated of root systems of type A, let us consider three vectors e_r, e_s and e_t whose nontrivial projections $\overline{e_r}$, $\overline{e_s}$ ans $\overline{e_t}$ are distincts and associated to three components Θ of type A_m, A_p and A_q. Let us consider the roots of the form $\alpha = \pm e_i \pm e_j$ and $\beta = \pm e_k \pm e_l$ and let us suppose their projections write

$$\overline{\alpha} = \pm(\overline{e_r} \pm \overline{e_s}) \qquad \text{and} \qquad \overline{\beta} = \pm(\overline{e_s} \pm \overline{e_t}) .$$

The projections are nontrivial, non-collinear, and non-orthogonal. The computations done in the previous subsection show that this family of vectors form a root system if and only if $m = p = q$. We also have in the projection of Σ the vectors of the form :

$$\overline{\gamma} = \pm\overline{e_v} \qquad \text{for } v \in \{r, s, t\}$$

Thus a system of type B_3. Furthermore, $m \geq 1$, we also have in the projection of Σ, vectors of the form :

$$\overline{\delta} = \pm 2\overline{e_v} \qquad \text{for } v \in \{r, s, t\}$$

and in the end we obtain a root system of type BC_3.

Let us consider now two roots $\alpha = \pm e_i \pm e_j$ and $\delta = \pm e_k$ whose projections write $\overline{\alpha} = \pm(\overline{e_r} \pm \overline{e_s})$ and $\overline{\delta} = \pm\overline{e_s}$. We observe that

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \qquad \text{and} \qquad ||\overline{\delta}||^2 = \frac{1}{p+1} \ .$$

Further $||\overline{\alpha}|| > ||\overline{\delta}||$ and we have :

$$\left(< \overline{\alpha}, \overline{\delta} >\right)^2 = \frac{1}{(p+1)^2} \ .$$

Therefore

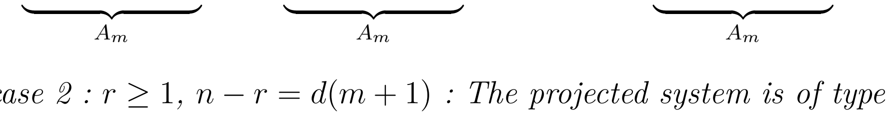
$$C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\delta})}\right)^2 = (1 + \frac{p+1}{m+1}) \qquad \text{and} \qquad R = \frac{||\overline{\alpha}||^2}{||\overline{\delta}||^2} = (1 + \frac{p+1}{m+1})$$

So we have $C = R$ which forces $C = R = 2$ and we recover the condition $m = p$.

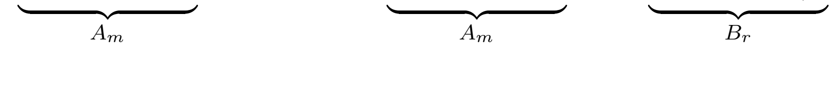
Let us also remark that two short roots (that is of type $\pm\overline{e_r}$) or long (that is of type $\pm 2\overline{e_r}$) (the length being relative to the length of roots $\pm(\overline{e_s} \pm \overline{e_t})$) are necessarily proportional or orthogonal. This observation exclude the occurence of a root system of type F_4. Combining these observations, we see that except if $m = 0$ (trivial case where the projection is the identity), we obtain maximal subsystems of type BC (in particular non reduced). Case 2 ($r \geq 1$) : $e_n \in \Theta$. The projection on the orthogonal complement of e_n gives a system B_{n-1} and reiterating this procedure when Θ contains B_r, we recover the case 1 previously treated for B_{n-r}. In conclusion, we have proven :

Lemma 90. *The maximal subsystems are of type B or BC. These contain the subsystems of type B, C or D of the same rank. Let us assume e_n belongs to a connected component of length r (then of type B_r), with $r \geq 0$ (the case $r = 0$ is the case in which e_n does not belong to Θ). Then, the set Σ_Θ contains a system of rank equal to the dimension d of $\mathfrak{a}_\Theta$ if the other components are all of the same length m (and type A_m), the intervals between any of these components being of length one with $n - r = (m+1)d$. The projected system is of type BC_d except if $m = 0$ in which case we obtain B_{n-r}.*

The case 1 : $r = 0$, $n = d(m+1)$: The projected system is of type BC_d if $m \geq 1$.



The case 2 : $r \geq 1$, $n - r = d(m+1)$: The projected system is of type BC_d.



This corresponds to a partition of the ordered basis Ξ of cardinal n in a union of $d+1$ ordered parts

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_{d+1}$$

where

$$\Xi_r = \{e_{(r-1)(m+1)+1} \cdots e_{r(m+1)}\} \qquad \textit{for } 1 \leq r \leq d \textit{ and } \qquad \Xi_{d+1} = \{e_{d(m+1)+1} \cdots e_{d(m+1)+r}\}$$

E.2.1. The case C_n

In this case the basis of a is formed with the e_i for $i \in \{1, \cdots, n\}$ and the elements of Σ are the $\pm 2e_i$ and the $\pm e_i \pm e_j$; Δ is constituted of the $\alpha_i = e_i - e_{i+1}$ for $i \leq n-1$ and of $\alpha_n = 2e_n$. The set Θ is an union of irreducible components all of type A except for at most one of type C_r. We distinguish two cases whether e_n belongs or not to Θ.

Case 1 $(r = 0)$: $2e_n \notin \Theta$. In this case Θ is an union of components of type A. As in the case of Σ of type A_n, let us consider three vectors e_r, e_s and e_t whose projections (which are non-zero) $\overline{e_r}$, $\overline{e_s}$ et $\overline{e_t}$ are distinct and associated to three components of Θ of type A_m, A_p and A_q and roots $\alpha = \pm e_i \pm e_j$ and $\beta = \pm e_k \pm e_l$ whose projections are

$$\overline{\alpha} = \pm(\overline{e_r} \pm \overline{e_s}) \qquad \text{and} \qquad \overline{\beta} = \pm(\overline{e_s} \pm \overline{e_t})$$

They will constitute a root system if and only if $m = p = q$. Then we obtain a root system of type C_3 constituted of the $\pm(\overline{e_r} \pm \overline{e_s})$, $\pm(\overline{e_s} \pm \overline{e_t})$, $\pm(\overline{e_r} \pm \overline{e_t})$ and $\pm 2\overline{e_v}$ for $v \in \{r, s, t\}$.

Let us now consider the two roots $\alpha = \pm e_i \pm e_j$ and $\beta = \pm 2e_k$ whose projections write

$$\overline{\alpha} = \pm\overline{e_r} \pm \overline{e_s} \qquad \text{and} \qquad \overline{\beta} = \pm\overline{2e_s} \; .$$

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \qquad \text{and} \qquad ||\overline{\beta}||^2 = \frac{4}{p+1}$$

and therefore

$$\left(< \overline{\alpha}, \overline{\beta} >\right)^2 = \frac{4}{(p+1)^2} \qquad \text{and} \qquad C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\beta})}\right)^2 = (1 + \frac{p+1}{m+1}) \; .$$

If we assume $||\overline{\beta}|| \geq ||\overline{\alpha}||$ we have

$$R = \frac{||\overline{\beta}||^2}{||\overline{\alpha}||^2} = \frac{4}{(1 + \frac{p+1}{m+1})}$$

and $CR = 4$. All the cases are *a priori* possible.

If $C = 2$ et $R = 2$ then we necessarily have $p = m$. The vectors $\overline{\alpha}$ and $\overline{\beta}$ are the basis of a root system of a type C_2 where $\overline{\beta}$ is the long root. The roots are

$\pm\overline{\alpha} = \pm(\overline{e_r}-\overline{e_s})$, $\pm\overline{\beta} = \pm 2\overline{e_s}$, $\pm(\overline{\alpha}+\overline{\beta}) = \pm(\overline{e_r}+\overline{e_s})$ and $\pm(2\overline{\alpha}+\overline{\beta}) = \pm 2\overline{e_r}$.

The case $C = 4$ and $R = 1$ implies

$$(p+1) = 3(m+1) \qquad \text{and therefore} \qquad p = 3m+2$$

Then $||\overline{\alpha}||$ and $||\overline{\beta}||$ constitute the basis of a a root system of type A_2 whose roots are

$$\pm\overline{\alpha} = \pm(\overline{e_r} - \overline{e_s}) \quad , \quad \pm\,\overline{\beta} = \pm 2\overline{e_s} \quad \text{and} \quad \pm\,(\overline{\alpha}+\overline{\beta}) = \pm(\overline{e_r}+\overline{e_s})$$

but the vector $\pm 2\overline{e_r}$ does not contribute to this system.
Finally if $C = 4/3$ we have

$$3(p+1) = (m+1) \qquad \text{and therefore} \qquad m = 3p+2$$

This forces $R = 3$ which is a configuration of simple roots for a root system of type G_2 where $\overline{\beta}$ is the long root. However, Σ_Θ does not contain all the necessary roots for such a system ; indeed the root

$$\overline{\beta} + 3\overline{\alpha} = 3\overline{e_r} - \overline{e_s}$$

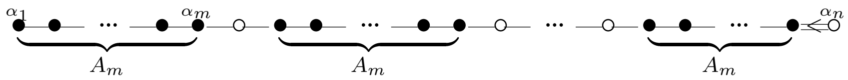
is not obtained.
Let us assume $||\overline{\alpha}|| \geq ||\overline{\beta}||$ we have $C/R = 4$ and we recover the case $C = 4$, $R = 1$ and therefore $(p+1) = 3(m+1)$.
Case 2 ($r \geq 1$) : $e_n \in \Theta$. The projection on the orthogonal complement of e_n gives a system of type BC_{n-1}. And, reiterating this procedure, we recover the case of BC_{n-r} which can be treated using our previous considerations on B_{n-r} and C_{n-r}.
To conclude, we have proved :

Lemma 91. *The maximal subsystems are of type A, B, C, D. Let us assume $2e_n$ belongs to a connected component of length r (and type C_r), with $r \geq 1$. The projection on the orthogonal of this component is a root system of type BC_{n-r}. We recover the case where $r = 0$, i.e where e_n does not belong to Θ for a system of type BC.*

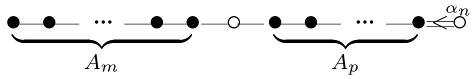
If $d \geq 3$ the set Σ_Θ contains a system of rank equal to the dimension d of $\mathfrak{a}_\Theta$ if the other components are all of the same length $m \geq 0$ (and type A_m), the intervals between any of these components being of length one with $n-r = (m+1)d$, then the projected system is of type BC_d (or C_n if $r = 0$ and $m = 0$, trivial case excluded).

If $d = 2$ we obtain either BC_d when the two components of type A are of length m or A_2 when $(p+1) = 3(m+1)$.

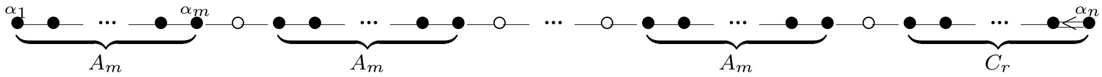
The case $r = 0$, with $n = (m+1)d$ and projected system C_d



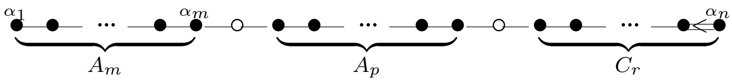
The case $r = 0$, with $p = 3m+2$ and $n = 4(m+1)$, and projected system containing A_2



The case $r \geq 1$, with $n - r = (m+1)d$ and projected system BC_d



The case $r \geq 1$, with $p = 3m+2$ and $n - r = 4(m+1)$, the projected system contains A_2



E.2.2. The case D_n

With the notations anologous to the previous cases the roots are the $\pm e_i \pm e_j$ and Δ is constituted of $\alpha_i = e_i - e_{i+1}$ for $i \leq n-1$ and of $\alpha_n = e_{n-1} + e_n$

Case 1 : $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$ are in Θ and the orthogonal complement of Θ admits the e_i for $1 \leq i \leq n-2$ as a basis. The projection on the orthogonal of e_n and e_{n-1} contain the $\pm e_i \pm e_j$ along with the roots $\pm e_i$ for i and j between 1 and $n-2$ obtained projecting the $\pm(e_i - e_n)$ We therefore obtain the system B_{n-2} already considered above.

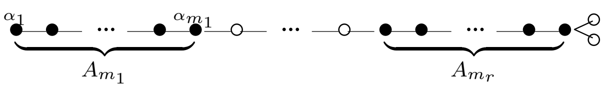
Case 2 : $\alpha_{n-1} = e_{n-1} - e_n$ is in Θ but $e_{n-1} + e_n$ is not. As in the case of root system of type B_n let us consider the three vectors e_r, e_s and e_t whose non-zero projections $\overline{e_r}$, $\overline{e_s}$ et $\overline{e_t}$ are distinct and associated to three components of Θ of type A_m, A_p and A_q. Once projected we find the $\pm\overline{e_r} \pm \overline{e_s}$ and $\pm\overline{e_s} \pm \overline{e_t}$. We also have

$$2\overline{e_r} = \overline{e_r} + \overline{e_{r+1}} = 2\overline{e_{r+1}}$$

if $\alpha_r = e_r - e_{r+1}$ belongs to a connected component of Θ. Therefore Σ_Θ contains a root system of type C_d if all the connected components of Θ are of the same cardinal m with $n = d(m+1)$.

Cas 2' : analogous to the case 2 when exchanging e_n with $-e_n$.

Cas 3 : Neither $\alpha_{n-1} = e_{n-1} - e_n$ nor $\alpha_n = e_{n-1} + e_n$ are in Θ.



We therefore have either an analogous situation to the one treated for A_n, or we consider $\overline{\alpha} = \pm\overline{e_{n-1}} \pm \overline{e_n}$ and $\overline{\beta} = \overline{e_s} \pm \overline{e_{n-1}}$.

In this case we have : $e_n = \overline{e_n}$ and therefore with the now familiar notations

$$R = \frac{(1+(p+1))}{(1+\frac{p+1}{m+1})} \qquad \text{and} \qquad C = (1+(p+1))(1+\frac{p+1}{m+1})$$

Therefore

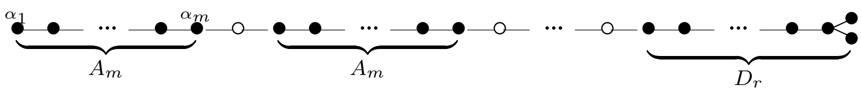
$$\frac{C}{R} = (1+\frac{p+1}{m+1})^2$$

which forces $R = 1$ and $C = 4$; thus $p = m = 0$. The existence of a system of maximal rank in the projection for a configuration of this sort forces $m_i = 0$ for any i, that is Θ is empty, a case which is possible but trivial hence excluded *a priori*.

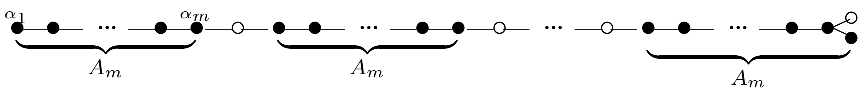
To sum up, we have proven the :

Lemma 92. *For a system of type D the subsystems in the projection are of type A, B, C or D. If $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$ are in Θ and if the others components of Θ are all of type A_m, the interval between two such components are of length one, with $n - r = (m+1)d$, then there exists a system of type BC_d in Σ_Θ. In the case 2 or 2', the projection contains a system of maximal rank of type C_d if all the components are of type A_m and if $n = (m+1)d$.*

The case 1 : $D_r \subset \Theta$ with $r \geq 2$; we recover the case of B_{n-r}.



The case 2 (or 2') : The projection contains a rank maximal system of type C_d if all the components are of type A_m and if $n = (m+1)d$.



E.3. The case of reducible Σ_Θ

We have seen that in order to obtain a projected root system irreducible and of maximal rank, we had to impose several constraints. Let us explain once more some of them. Let us first consider two components A_m and A_q of Θ, let e_r and e_s be the vectors in the basis vectors of smallest index such $\Xi_r = \{e_r, \ldots, e_{r+m}\}$ corresponds to A_m and Ξ_s to A_q. Let us assume two simple consecutive roots α_{k-1} and α_k are outside of Θ and $k = r + m + 1 = s - 1$. Then $\Xi_k = \{e_k\}$. Let us consider the projections of α_{k-1} and α_k : Since e_k is orthogonal to all roots in Θ,

$\overline{e_k} = e_k$.
Therefore :

$$||\overline{\alpha_{k-1}}||^2 = ||\overline{e_{k-1}} - \overline{e_k}||^2 = \frac{1}{m+1} + 1 \ .$$

$$||\overline{\alpha_k}||^2 = ||\overline{e_k} - \overline{e_{k+1}}||^2 = 1 + \frac{1}{q+1} \ .$$

$$(< \overline{\alpha_{k-1}}, \overline{\alpha_k} >)^2 = 1 \ .$$

Then

$$C = \left(\frac{1}{\cos(\overline{\alpha_{k-1}}, \overline{\alpha_k})}\right)^2 = (\frac{1}{m+1} + 1)(1 + \frac{1}{q+1})$$

and if we assume $||\overline{\alpha_{k-1}}|| \geq ||\overline{\alpha_k}||$ i.e $m \geq q$, we have :

$$R = \frac{||\overline{\alpha_{k-1}}||^2}{||\overline{\alpha_k}||^2} = \frac{\frac{1}{m+1} + 1}{\left(1 + \frac{1}{q+1}\right)}$$

If α_k and α_{k-1} were to be part of a root system, we would need

$$\frac{C}{R} = \left(1 + \frac{1}{m+1}\right)^2 \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = \frac{1}{\left(1 + \frac{1}{q+1}\right)}^2 = 4 \ .$$

This implies $m = 0$ and $\left(1 + \frac{1}{q+1}\right) = 1/4$ a contradiction. This illustrates the fact that in the main theorem (Theorem 88) the intervals between the irreducible connected components of Θ need to be of length one, and *at most one*.

Let us now observe that another possibility would be to obtain a reducible root system such as $A_1 \times A_1 \times \ldots \times A_1$. This case is not excluded but it would not be possible to find such a system of *maximal rank*.

Indeed, by the formulas obtained for the case of Σ of type A for instance, we had :

$$\left(< \overline{\alpha}, \overline{\beta} >\right)^2 = \frac{1}{(p+1)^2} \ .$$

This excludes the possibility of α and β being orthogonal. Therefore for two consecutive roots in the projection (projections of simple roots), it is not possible to obtain a system of type $A_1 \times A_1$.

If there is a sequence of connected consecutive components of Θ of type A that we index by an integer i (in increasing order) and length q_i with $q_i \neq q_{i+1}$ for any i, let us denote $\overline{\alpha_i} = \overline{e_r} - \overline{e_s}$ where $e_r \in A_{q_i}$ and $e_s \in A_{q_{i+1}}$.

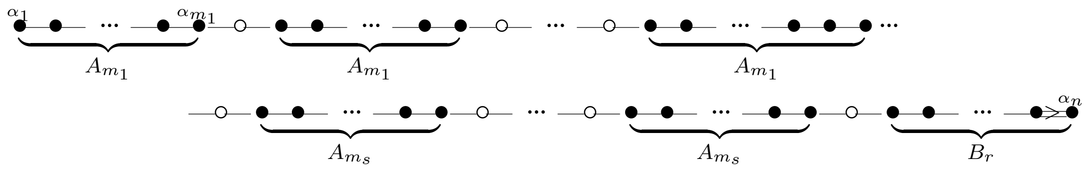
Further, let us denote $\overline{\alpha_{i+2}} = \overline{e_t} - \overline{e_z}$ where $e_t \in A_{q_{i+2}}$ and $e_z \in A_{q_{i+3}}$. The orthogonal roots $\overline{\alpha_i}$ and $\overline{\alpha_{i+2}}$ form a root system of type $A_1 \times A_1$. The root $\overline{\alpha_{i+1}} = \overline{e_s} - \overline{e_t}$ does not contribute to this subsystem.

Therefore, the maximal number of A_1 factor such that the reducible root system $A_1 \times A_1$ appear in Σ_Θ is $d/2$. By a similar reasoning, it would be possible to obtain a reducible system of type $A_2 \times A_2 \times \ldots A_2$ if Θ is composed of a succession of connected components of type A such that the three first ones are of length m, the three next ones of length $q \neq m$ ..etc. Then the projection of the root connecting A_m and A_q would not contribute to this subsystem.

In general, the contrapositive of the main theorem (Theorem 88) is that if the irreducible components of Θ of type A are *not all of the same length*- the interval between two consecutive of them still of length one- then Σ_Θ contains many irreducible components ($\Sigma_{\Theta,i}$). The number of such irreducible components is as many as there are *changes of length* plus one.

That is, if there are d_1 components of type A_{m_1}, followed by d_2 components of A_{m_2}, et cetera until d_s components of A_{m_s}, such that $m_i \neq m_{i+1}$ for any i, and one last component of type B or C or D, they are $s-1$ changes in the length (m_i) and therefore s irreducible connected components in Σ_Θ.

Let us illustrate such cases with a Dynkin diagram of Θ of type B :



F. Bala-Carter theory

In this section, we discuss unipotent conjugacy classes in a connected reductive complex algebraic group. The discussion can be reduced to the case in which G is semi-simple since the natural homomorphism from G to G/Z_G induces a bijection between unipotent conjugacy classes of G and those of G/Z_G (Proposition 5.1.1 in Carter 1985).

Using a further bijection between unipotent conjugacy classes of G and nilpotent $Ad(G)$-orbits on the Lie algebra $\mathfrak{g}$ (A theorem of Springer- Steinberg, see al 1970), we will explain the classification of the latter.

So let G be a semi-simple adjoint group over $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra over $\mathbb{C}$. It is well-known that if $\mathfrak{g}$ is semi-simple then a Cartan subalgebra $\mathfrak{t}$ is commutative, and $\mathfrak{g}$ is completely reducible under $\mathfrak{t}$, acting by the adjoint representation[see « Lie Groups and Linear Algebraic Groups I »]. We can consider $\Phi_0 = \Phi(\mathfrak{t}; \mathfrak{g})$ the roots of $\mathfrak{t}$ in $\mathfrak{g}$, Φ_0^+ the corresponding set of positive roots, and $\Delta \subset \Phi_0$ a set of simple roots.

There is a decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \overline{\mathfrak{n}}$, where $\overline{\mathfrak{n}}$ is the nilpotent radical of the Borel subalgebra opposite to $\mathfrak{b}$.

Let $\mathcal{N} = \mathcal{N}_\mathfrak{g}$ be the cone of nilpotent elements in $\mathfrak{g}$. This cone is the disjoint union of a finite number of G- orbits. In the 1950's different parametrizations of the set of nilpotent G-orbits in $\mathfrak{g}$, $G\backslash\mathcal{N}$ were proposed : partition-type classifications and weighted Dynkin diagrams, we will discuss the second.

F.1. Weigthed Dynkin diagrams

Let $\mathcal{O}$ be a nilpotent orbit in $G\backslash\mathcal{N}$ and let $x \in \mathcal{O}$ be a representative element. A theorem of Jacobson- Morozov extends x to a standard ($\mathfrak{sl}_2$) triple $\{x, h, y\} \in \mathfrak{g}$, where h can be chosen to lie in the fundamental dominant Weyl chamber :

$$\{h' \in \mathfrak{g} | \text{Re}(\alpha(h')) \geq 0, \forall \alpha \in \Delta \text{ and whenever } \text{Re}(\alpha(h')) = 0, \text{Im}(\alpha(h')) \geq 0\}$$

Theorem 93 (Kostant,Kostant 1959). *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. A nilpotent orbit $\mathcal{O}$ is completely determined by the values $[\alpha_1(h), \alpha_2(h), \ldots, \alpha_n(h)]$.*

For every simple root α in Δ, we have $\langle \alpha, h \rangle \in \{0, 1, 2\}$ (see section 3.5 in Collingwood et McGovern 1993).

If we label every node of the Dynkin diagram of $\mathfrak{g}$ with the eigenvalues $\alpha(h) = \langle \alpha, h \rangle$ of h on the corresponding simple root space $\mathfrak{g}_\alpha$, then all labels are 0,1 or 2. We call such a labeled Dynkin diagram, **a weighted Dynkin diagram**.

F.2. The Bala-Carter classification

The drawback of partition-type classifications was that they only apply to classical Lie algebras whereas a « good » parametrization of nilpotent orbits should be applicable to any semisimple Lie algebra. In two seminal papers (Bala et Carter 1976a, Bala et Carter 1976b), appearing in 1976, Bala and Carter achieved such parametrization.

The key notion used by Bala and Carter was the notion of distinguished nilpotent element. It is an element that is not contained in any proper Levi subalgebra. Alternatively, a nilpotent element $n \in \mathfrak{g}$ is called distinguished if it does not commute with any non-zero semi-simple element of $\mathfrak{g}$. Or also, a nilpotent element X (resp. orbit $\mathcal{O}_X$) is distinguished if the only Levi subalgebra containing X (resp. meeting $\mathcal{O}_X$) is $\mathfrak{g}$ itself.

By focusing on the special properties of the orbits of distinguished elements in Levi subalgebras they could eventually parametrize all nilpotent orbits in $\mathfrak{g}$.

We now need to introduce the definition of distinguished parabolic subgroup and distinguished parabolic subalgebra.

Definition 94 (distinguished parabolic subgroup)**.** Let P_J be a standard parabolic subgroup of G a group of adjoint type, with Levi decomposition $P_J = N_J L_J$. The Levi subgroup L_J decomposes as $L'_J Z(L_J)$ where L'_J is semisimple and $Z(L_J)$ is a torus.

The parabolic subgroup P_J is defined to be distinguished provided $\dim L_J = \dim N_J/N'_J$ [8]

Definition 95 (distinguished parabolic subalgebra)**.** A parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ is called distinguished if $\dim \mathfrak{l} = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$, in which $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition of $\mathfrak{p}$, with Levi part $\mathfrak{l}$.

The main theorem is the following :

Theorem 96 (5.9.5 in Carter 1985)**.** *Let G be a simple algebraic group of adjoint type over F. Suppose the characteristic p of F is either zero, or $p > 3(h-1)$ where h is the Coxeter number of G. Let $\mathfrak{g}$ be the Lie algebra of G. Then :*

1. *There is a bijective map between the G-orbits of distinguished nilpotent elements of $\mathfrak{g}$ and the conjugacy classes of the distinguished parabolic subgroups of G. The G-orbit corresponding to a given parabolic subgroup P contains the dense orbit of P acting on the Lie algebra of its unipotent radical.*
2. *There is a bijective map between the G-orbits of nilpotent elements of $\mathfrak{g}$ and the G-classes of pairs $(L, P_{L'})$ where L is a Levi subgroup of G and $P_{L'}$ a distinguished parabolic subgroup of the semi-simple part L' of L. The G-orbit*

8. For a subset $J \subseteq \Delta$, one defines a function $\eta_J : \Phi_0 \to 2\mathbb{Z}$ which equals 0 on any root in Δ_J and 2 for any root in $\Delta - \Delta_J$, then $N'_J = \prod_{\eta_J(\alpha)>2} N_\alpha$, N_α is the root subgroup corresponding to the root α. See Section 5.8 in Carter 1985

corresponding to a given pair $(L, P_{L'})$ contains the dense orbit of $P_{L'}$ acting on the Lie algebra of its unipotent radical.

In term of Lie algebras, we have the following one-to-one correspondences :

$$\begin{Bmatrix} \text{Distinguished nil-} \\ \text{potent Ad}(G)\text{-orbits} \\ \text{of } \mathfrak{g} \end{Bmatrix} \leftrightarrow \begin{Bmatrix} G \text{ conjugacy classes of} \\ \text{distinguished parabolic} \\ \text{subalgebras of } \mathfrak{g} \end{Bmatrix} \tag{1.39}$$

$$\begin{Bmatrix} \text{Nilpotent Ad}(G)\text{-orbits} \\ \text{of } \mathfrak{g} \end{Bmatrix} \leftrightarrow \begin{Bmatrix} G \text{ conjugacy classes of} \\ \text{pairs } (\mathfrak{p}, \mathfrak{m}) \text{ of } \mathfrak{g} \end{Bmatrix} \tag{1.40}$$

in which $\mathfrak{m}$ is a Levi factor, $\mathfrak{p} \subseteq \mathfrak{m}'$ is a distinguished parabolic subalgebra of the semi-simple part of $\mathfrak{m}$.

We sketch the ideas behind these correspondences.

As above, given a non-zero nilpotent element in $\mathfrak{g}$, let $\{e, h, f\}$ denote the standards basis of the $\mathfrak{sl}_2$ Lie algebra. The Jacobson-Morozov Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ satisfies $\phi(e) = n \in \mathfrak{n}$ and $\phi(h) = \gamma$ is in the dominant chamber of $\mathfrak{t}$.

The adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ yields a grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ in which

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} | ad(\gamma)(x) = ix\} \, ; [\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i + j)$$

and $n \in \mathfrak{g}(2)$. Further, set

$$\begin{cases} \mathfrak{p} = \mathfrak{p}(\gamma) = \oplus_{i\geq 0}\mathfrak{g}(i) \\ \mathfrak{u} = \oplus_{i>0}\mathfrak{g}(i) \\ \mathfrak{l} = \mathfrak{g}(0) \end{cases} \tag{1.41}$$

The Lie subalgebra $\mathfrak{p}$ contains $\mathfrak{b}$, and is thus a parabolic subalgebra whose Levi decomposition is $\mathfrak{p} = \mathfrak{u} \oplus \mathfrak{l}$.

On the other hand, starting with a subset $J \subseteq \Delta$, and denoting $\mathfrak{p}_J$ the standard parabolic subalgebra, one defines a function $\eta_J : \Phi_0 \to \mathbb{Z}$, defined on roots of Δ as twice the indicator function of J and extended linearly to all roots.

We obtain a grading : $\mathfrak{g} = \oplus_{i\geq 0}\mathfrak{g}_J(i)$ by declaring $\mathfrak{g}_J(0) = \mathfrak{t} \oplus \sum_{\eta_J(\alpha)=0} \mathfrak{g}_\alpha$ and otherwise $\mathfrak{g}_J(i) = \sum_{\eta_J(\alpha)=i} \mathfrak{g}_\alpha$. Then, $\mathfrak{p}_J = \oplus_{i\geq 0}\mathfrak{g}_J(i)$ and its nilpotent radical is $\mathfrak{n}_J = \oplus_{i>0}\mathfrak{g}_J(i)$.

To summarize, to the standard triple containing n one attaches a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$.

If $\dim \mathfrak{g}(1) = 0$, then we call n (resp. $\mathcal{O}_n$) an even nilpotent element (even nilpotent orbit, respectively).

Proposition 97 (Corollary 3.8.8 in Collingwood et McGovern 1993). *A weighted Dynkin diagram has labels 0 or 2 if and only if it corresponds to an even nilpotent orbit (i.e, if $\dim \mathfrak{g}(1) = 0$)*

Proposition 98. *The standard parabolic subalgebra $\mathfrak{p}_J$ is distinguished if and only if* $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$. *In this case, if n is any element in the unique open orbit of the parabolic subgroup P_J on its nilpotent radical $\mathfrak{n}_J$, then the parabolic subalgebra associated to n as in (1.41) equals $\mathfrak{p}_J$.*

A distinguished nilpotent element also satisfies the following :

Proposition 99. *A nilpotent element $n \in \mathfrak{g}$ is distinguished if and only if* $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$. *Moreover, if $n \in \mathfrak{g}$ is distinguished, then* $\dim \mathfrak{g}(1) = 0$.

Theorem 100 (Theorem 8.2.3 in Collingwood et McGovern 1993)**.** *Any distinguished orbit in $\mathfrak{g}$ is even.*

Theorem 101 (Theorem 8.2.14 in Collingwood et McGovern 1993)**.**

1. *If $\mathfrak{g}$ is of type A, then the only distinguished orbit is principal.*
2. *If $\mathfrak{g}$ is of type B, C or D, then an orbit is distinguished if and only if its partition has no repeated parts. Thus the partition of a distinguished orbit in types B, D has only odd parts, each occurring once, while the partition of a distinguished orbit in type C has only even parts, each occurring once.*

We can now write the correspondences :
Pick a distinguished element n. By Proposition 99, $\mathfrak{p}$ is a standard parabolic subalgebra $\mathfrak{p}_J$ for $J = \{\alpha \in \Delta | \mathfrak{g}_\alpha \subseteq \mathfrak{g}(2)\}$ which is distinguished by Proposition 98, and we obtain the map inducing the first bijective correspondence : $n \to \mathfrak{p}$

By Proposition 97, since we are given this distinguished parabolic algebra $\mathfrak{p}$, $\gamma = \phi(h)$ is an even Weighted Dynkin Diagram for the semi-simple Lie algebra $\mathfrak{g}$.

For the second, one can choose a minimal Levi subalgebra $\mathfrak{m}$ containing n (cf Prop 5.9.3 in Carter 1985) which modulo conjugation, can be assumed to be a Levi factor of a parabolic subalgebra containing $\mathfrak{b}$. By minimality of $\mathfrak{m}$, it follows that $n \in \mathfrak{m}' = [\mathfrak{m}, \mathfrak{m}]$ is a distinguished nilpotent element in $\mathfrak{m}'$, and then by Proposition 98, there is a distinguished parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{m}'$ corresponding to n. One can construct a map induced by $n \to (\mathfrak{m}, \mathfrak{p})$. On the other direction, one associates to a conjugacy class of the pair $(\mathfrak{m}, \mathfrak{p})$ the orbit Ad$(G)n$ in which $n \in \mathfrak{n}_\mathfrak{p}$ is any element in the unique dense adjoint orbit of P on $\mathfrak{n}_\mathfrak{p}$, with the latter being the nilradical of $\mathfrak{p}$ and P the parabolic subgroup of G associated to $\mathfrak{p}$.

F.3. Distinguished Nilpotent orbits and residual points

The connection with the notion of residual point is now made accessible.
Let G be a Chevalley (semi-simple) group and $T \subseteq B$ a maximal split torus and a Borel subgroup. We have a root datum $\mathcal{R}(G, B, T)$. By reversing the role of $X^*(T)$ and $X_*(T)$, we obtain a new root datum $\mathcal{R}^\vee = (X_*(T), \Delta, X^*(T), \Delta^\vee)$. Let $({}^LG, {}^LB, {}^LT)$ be the triple with root datum $\mathcal{R}^\vee$. The L-group ${}^LG$ is the dual group, with maximal torus ${}^LT$, and Borel subgroup ${}^LB$. Denote the respective

Lie algebra ${}^L\mathfrak{g}$, ${}^L\mathfrak{t}$ and ${}^L\mathfrak{b}$. Let $(V^*, \langle , \rangle)$ be a finite dimensional Euclidean space containing and spanned by the root system : $\Delta \subseteq V^*$, the canonical pairing between V and V^* is denoted by $\langle , \rangle$. We fix an inner product on V by transport of structure from $(V^*, \langle , \rangle)$ via the canonical isomorphism $V^* \to V$ associated with $\langle , \rangle$. Thus this map becomes an isometry, and for each $\alpha \in \Delta$, the coroot $\check{\alpha} \in V$ is given as the image of $2 \langle \alpha, \alpha \rangle^{-1} \alpha \in V^*$.

To this data we associate the Weyl group W_0 generated by the reflexions s_α $(s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha$ and $s_\alpha(y) = y - \langle \alpha, y \rangle \check{\alpha})$ over the hyperplanes $H_\alpha \subseteq V^*$ consisting of elements $x \in V^*$ which are orthogonal to $\check{\alpha}$ with respect to $\langle , \rangle$.

Let us make a remark before stating the correspondence result related to our use in this manuscript :

Remark 12. The bijective correspondence (below) is originally formulated for residual subspaces. Let k be the « coupling parameter » as defined in Heckman et Opdam 1997. An affine subspace $L \subseteq V$ is called residual if, for a root system Φ (in a root datum)

$$\# \{\alpha \in \Phi | \langle \alpha, L \rangle = k\} = \# \{\alpha \in \Phi | \langle \alpha, L \rangle = 0\} + \operatorname{codim} L$$

(If $\mathcal{R}$ is semi-simple, there exist residual subspace which are singletons $\{\lambda\} \subseteq V$, the residual points).

For example, when the parameter k (called « coupling parameter » in Heckman et Opdam 1997) equals 1, the Weyl vector $\rho = \frac{1}{2} \sum_{\alpha \in \Phi} \alpha$ is a residual point, since the above equation is verified. More generally, for any $k = (k_\alpha)_{\alpha \in \Phi}$, the vector $\rho(k) = \frac{1}{2} \sum_{\alpha \in \Phi} k_\alpha \alpha$ is a residual point.

Then the bijective correspondence is given between the set of nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0- orbits of residual subspaces.

We mention the following result partially related to Proposition 24. The bijective correspondence concerns only unramified characters and we fix the parameter $k_\alpha = 1$ for all $\alpha \in \Phi_0$.

Proposition 102. *There is a bijective correspondence $\mathcal{O}_{W_0\lambda(\mathcal{O})} \leftrightarrow W_0\lambda(\mathcal{O})$ between the set of distinguished nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0-orbits of residual points.*

Proof. This particular bijection is a specific case of the larger bijective correspondence given between the set of nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0-orbits of residual subspaces. It is discussed in details in [Opdam 2004, Appendices A and B], but also in [Heiermann 2006, Proposition 6.2]. □

Let $({}^L\mathfrak{m}, {}^L\mathfrak{p})$ be a representative of a class, for which ${}^L\mathfrak{m} = {}^L\mathfrak{g}$ and ${}^L\mathfrak{p} \subseteq {}^L\mathfrak{g}$ is a standard distinguished parabolic subalgebra. We have a corresponding distinguished nilpotent orbit $\mathcal{O}$. With Proposition 98, the data ${}^L\mathfrak{p}$ is equivalent to the assignement of an even weighted Dynkin diagram : $2\lambda(\mathcal{O})$.

Since we have $\dim \mathfrak{g}(0) = \dim \mathfrak{h} + \# \{\alpha \in \Phi | \langle \check{\alpha}, 2\lambda(\mathcal{O})\rangle = 0\}$ and

$$\dim \mathfrak{g}(2) = \# \{\alpha \in \Phi | \langle \check{\alpha}, 2\lambda(\mathcal{O})\rangle = 2\}$$

The assignement of an even weighted Dynkin diagram implies $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$ and this equality sets $\lambda(\mathcal{O})$ as a residual point.

The definition of $\lambda(\mathcal{O})$ depends on the choice of positive roots and Borel subgroup ${}^{L}B$. A different choice yields a different element on the same W_0-orbit.

For the sake of completeness, we quote the proposition as given in [Opdam 2004, Appendices A and B] :

Proposition 103 (Proposition 8.1 in Opdam 2004)**.** *(i) If r is a residual point with polar decomposition $r = sc = sexp(\gamma) \in T_uT_{rs}$ and γ is dominant, then the centralizer $C_{\mathfrak{g}}(s)$ of s in $\mathfrak{g} := Lie(G)$ is a semi-simple subalgebra of $\mathfrak{g}$ of rank equal to rank($\mathfrak{g}$), and γ/k is the weighted Dynkin diagrams (confer page 175 of* Carter *1985) of a distinguished nilpotent class of $C_{\mathfrak{g}}(s)$.*

(ii) Conversely, let $s \in T_u$ be such that the centralizer algebra $C_{\mathfrak{g}}(s)$ is semisimple and let $e \in C_{\mathfrak{g}}(s)$ be a distinguished nilpotent element. If h denotes the weighted Dynkin diagram of e then $r = sc$ with $c := exp(kh)$ is a residual point.

(iii) The above maps define a 1 - 1 correspondence between W_0-orbits of residual points on the one hand, and conjugacy classes of pairs (s, e) with $s \in G$ semisimple such that $C_{\mathfrak{g}}(s)$ is semisimple, and e a distinguished nilpotent element in $C_{\mathfrak{g}}(s)$.

(iv) Likewise there is a $1 - 1$ correspondence between W_0-orbits of residual points and conjugacy classes of pairs (s, u) with $C_G(s)$ semisimple and u a distinguished nilpotent element of $C_G(s)^0$.

G. Characterization of cuspidal supports giving minimal Langlands parameters for the order on elements (ν', P') with P' maximal parabolic subgroup

We have observed that adding a linear residual segment $(a, \dots, -a_{-})$ to a residual segment $(\underline{n})$ such that $a = 2a+1, a = 2a_{-}+1$ are not in the Jordan block of the discrete series τ attached to $(\underline{n})$ (or equivalently a and a_{-} are not in the Jumps set associated to $(\underline{n})$) yields a new residual segment $(\underline{n'})$.

In this section, we characterize the linear residual segment $(a, \dots, b)$ such that the element $(a, \dots, b)(\underline{n})$ is not in the W_σ-orbit of a residual segment, and the Langlands parameter $\frac{a+b}{2}$ is minimal for the order given in Lemma 31.

We introduce the notion of unalterability of the cuspidal support (see Definition 105 below). We apply this notion on cuspidal strings $(a, b, \underline{n})$.

In this section, we abandon the notation $(\underline{n})$ for residual segments and denote them $(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}} \dots 1^{n_1}0^{n_0})$, where n_i is the number of times the (half)-integer i is repeated in the residual segment.

Let us explain how we study a given W_σ-cuspidal string associated to a maximal parabolic subgroup. On the W_σ-cuspidal string, we minimize a potential which is the Langlands parameter : $\frac{a+b}{2}$.

To do so, starting from any point in this set :

$$(a, \dots, b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

with $a > b$ we can do the following actions :

Addition If $a \geq (\ell+m+1)$, one can add *all elements* $(a, \dots, (\ell+m+1))$ in the right hand residual segment. We call this process *addition*.

Insertion One can also add *only some elements* from left hand residual segment within the right hand residual segment.

Removal One can remove elements from the right hand residual segment to enlarge the left hand residual segment on its right end. In particular, negative elements can be added on the right of the right hand residual segment.

Consider again some initial point in this set :

$$(a, \dots, b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

If $a \geq (\ell+m+1)$, then we add all elements from $(\ell+m+1)$ to a at the left end of the right hand residual segment. Then we have

$$(a_2, \dots, b)(a, \dots, (\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

where $a_2 \leq (\ell+m)$

Either we can insert all elements $(a_2, \ldots, b)$, in which case we reach a residual segment, or we cannot.

One can insert elements starting from a_2 and decreasing to b, or one can remove elements on the left hand segment to add at the left end of the right hand residual segment. With these two procedures, one obtains a point

$$(a', \ldots, b')(\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

where $a' \leq a_2 \leq a$ or $b' \leq b$.

We reach a minimal Langlands parameter $\nu_{min} = \frac{a'+b'}{2}$. This reduction procedure yields an *unalterable* cuspidal string (A precise definition is given in 105)

$$(a', \ldots, b')(\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

We will describe the results for root systems of type B_l, C_l, D_l below, but we first explain how one compares two Langlands parameters :

Let $(a, b, \underline{n})$ be a cuspidal string whose associated discrete series is $St_q|.|^a \times \pi$ The integer a defining the twist of the discrete series is :
$a = a - \frac{a-b}{2} = \frac{a+b}{2} \geq 0$ and the length of the segment is $q = a - b + 1$.

Therefore the Langlands parameter corresponding to this cuspidal string takes the form :

$$\nu = (\underbrace{a, a, \ldots,}_{q \text{ times}} 0, 0, \ldots 0)$$

Lemma 104. *Let π_1, π_2 be generic discrete series representations of a maximal standard Levi subgroup M of a group G of type B, C or D, with same cuspidal support. Let (P, π_1, λ_1) and (P, π_2, λ_2) be Langlands data, with λ_1, λ_2 Langlands parameters. Let the cuspidal string associated to π_1 (resp. π_2) be $(a_1, b_1, \underline{n_1})$ (resp. $(a_2, b_2, \underline{n_2})$). Then $\lambda_1 \geq \lambda_2$ for the order given in Lemma 31 if and only if :*

$$\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2} \geq 0$$

and

$$\frac{a_1+b_1}{2}(a_1 - b_1 + 1) - \frac{a_2+b_2}{2}(a_2 - b_2 + 1) \geq 0$$

Proof. From the classical theory of segments the Langlands parameter given by this cuspidal support is : Since $a_1 - \frac{a_1-b_1}{2} = \frac{a_1+b_1}{2}$

$$\lambda_1 = (\underbrace{\frac{a_1+b_1}{2}, \frac{a_1+b_1}{2}, \ldots,}_{q \text{ times}} 0, 0, \ldots 0) \qquad \in a^*_{M_{\text{cusp}}}$$

where the length q of the left hand segment is $a_1 - b_1 + 1$.

$$\lambda_2 = (\underbrace{\frac{a_2+b_2}{2}, \frac{a_2+b_2}{2}, \dots,}_{q' \text{ times}} 0, 0, \dots 0) \qquad \in a^*_{M_{\text{cusp}}}$$

$q' = a_2 - b_2 + 1$.

$$\lambda_1 - \lambda_2 = \begin{cases} (\underbrace{\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}, \frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}, \dots,}_{q' \text{ times}} \underbrace{\frac{a_1+b_1}{2}, \frac{a_1+b_1}{2}, \dots}_{q-q' \text{ times}} 0, 0, \dots 0) & \text{if } q' \le q \\ (\underbrace{\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}, \frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}, \dots,}_{q \text{ times}} \underbrace{-\frac{a_2+b_2}{2}, -\frac{a_2+b_2}{2}, \dots}_{q'-q \text{ times}} 0, 0, \dots 0) & \text{if } q \le q' \end{cases}$$

If we denote $\lambda_1 - \lambda_2 = (a_1, a_2, \dots, a_q, \dots a_{q'}, 0, ..., 0)$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$, let us write $x_i = \sum_k = 1^i a_k$.

If $q' \le q$

$$x_1 = \frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}$$

$$x_2 = 2(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2})$$

$$x_{q'} = q'(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2})$$

$$x_q = q'(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}) + (q-q')\frac{a_1+b_1}{2}$$

and for any $i < q$, $x_i \ge x_q$

If $q \le q'$ Notice that

$$x_1 = \frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}$$

$$x_2 = 2(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2})$$

$$x_q = q(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2})$$

$$x_{q'} = q(\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}) + (q'-q)(-(\frac{a_2+b_2}{2})) = q(\frac{a_1+b_1}{2}) + (q'-q+q)(-(\frac{a_2+b_2}{2}))$$

$$= q(\frac{a_1+b_1}{2}) + (q')(-(\frac{a_2+b_2}{2}))$$

and for any $i < q'$, $x_i \ge x_{q'}$. From the Definition 30 and the Lemma 32, $\lambda_1 \ge \lambda_2 \Leftrightarrow \lambda_1 - \lambda_2 \ge 0$ translates in the requirement that all x_i's need to be be positive.

The result follows. $\square$

Definition 105. Let (P, τ, ν) be Langlands data with τ irreducible discrete series and P maximal. The cuspidal support (σ, λ), with σ unitary cuspidal representation, of τ_ν is said to be *unalterable* if there does not exist a Langlands data (P', τ', ν') with P' maximal, such that σ is an element of the cuspidal support of the corresponding standard module, $I_{P'}^G(\tau'_{\nu'})$ and $\nu' < \nu$.

Remark 13. Once the cuspidal unitary representation σ is fixed (and we consider the W_σ-orbit of σ_λ), the unalterability condition *is charactezized on the parameter* of σ, so we will speak of *unalterable cuspidal strings.*

G.1. B_l

Consider the two sequences of integers : $(a, a-1, \dots b)(\underline{n})$, where the tuple $\underline{n}$ satisfies the following conditions : $n_\ell = 1, n_{\ell-1} = 2, n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$ and $n_0 = \frac{n_1 - 1}{2}$ or $\frac{n_1}{2}$.

In the following lemmas, we will characterize the form of the cuspidal strings corresponding to the minimal Langlands parameters under certain constraints satisfied by the values of the n_i's as just given.

Remark 14. Notice first that it is enough to consider the case of $m = 0$. That is if $m \geq 2$, and $a = (\ell+1)$, the elements $((\ell+m)(\ell+m-1)\dots(\ell+2))$ can be assumed to lie at the left-end of the segment $(a, a-1, \dots, b)$.

If $a = \ell$ and $m \geq 1$, the elements $((\ell+m)(\ell+m-1)\dots(\ell+1))$ will be put at the left-end of the segment $(a, a-1, \dots, b)$. A configuration of type

$$(a, a-1, \dots b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

will be studied as

$$((\ell+m)(\ell+m-1)\dots, a, a-1, \dots b)(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

Lemma 106. *The cuspidal string given by $(a, a-1, \dots b)(\underline{n})$ is unalterable if and only if :*

1. *When $n_{i-1} = n_i + 1$ for all i and $n_0 = \frac{n_1-1}{2}$, $-\ell \leq b \leq -1$ and $b \leq a \leq \ell$, or $b = 1$ and $(\ell+1) > a > b$.*
2. *When $n_{i-1} = n_i + 1$ for all i and $n_0 = \frac{n_1}{2}$; $\ell \leq b \leq 0$ and $b \leq a \leq \ell$ and if $a = b = 1$.*

Proof. 1. — If $b = 0, b \leq a \leq (\ell+m)$; we reduce by the above remark 14 to the case $a = \ell+1, b = 0$ and get a residual point :

$$((\ell+1)^{n_\ell}(\ell)^{n_{\ell-1}}(\ell-1)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1+1}0^{n_0+1})$$

— If $\mathscr{b} = 0, \mathscr{b} \leq \mathscr{a} \leq \ell$, one withdraws from the right

$$(\mathscr{a}, \ldots 0 \ldots - \mathscr{a})((\ell-1)^{n_\ell}(\ell-2)^{n_{\ell-1}} \ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

or

$$(\mathscr{a}, \ldots 0 \ldots - (\mathscr{a}-1))((\ell-1)^{n_\ell}(\ell-2)^{n_{\ell-1}} \ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

with changed values of the n_i's, i.e. $n'_i \neq n_i$ for some i, $n'_1 = n_1 - 1$ and $n'_0 = 1/2(n'_1)$.

— If $\mathscr{b} = 1, \mathscr{a} = \ell+1, (\mathscr{a}, \ldots, \mathscr{b})(\ell^{n_\ell} \ldots 0^{n_0})$ transforms into $(1)((\ell+1)^{n_\ell}\ell^{n_{\ell-1}} \ldots (\ell-i)^{n_{\ell-i}+1} \ldots 1^{n_1}0^{n_0})$

— If $\mathscr{b} = 1, (\ell+1) > \mathscr{a} > \mathscr{b}$, we get an unalterable cuspidal support : $(\mathscr{a}, \mathscr{a}-1, \ldots \mathscr{b})(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$ giving minimal Langlands parameters. Indeed, the cuspidal support cannot be modified (under the constraint of having a maximal parabolic subgroup) without withdrawing a zero on the right hand residual segment and therefore two ones. But on the left, only one 1 can be added.

Similarly, if $m \geq 2$, and $\mathscr{a} = \ell + m, \mathscr{b} = 1$, then we get unalterable cuspidal string :

$$(\mathscr{a} = \ell+m, \mathscr{a}-1, \ldots, \mathscr{b})((\ell+m)(\ell+m-1) \ldots ((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$$

— Else if $0 \geq \mathscr{b} \geq -(\ell-1)$ and $\mathscr{a} = \ell$; one gets a point $(\ell(\ell-1) \ldots 0 \ldots \mathscr{b})(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$ with minimal Langlands parameter.

— If $\mathscr{b} > 1$ and $\mathscr{a} = \ell$ or $\mathscr{a} = \ell+1$; one withdraws the zeroes and subsequent higher numbers to get :

$$(\ell(\ell-1) \ldots -\ell)((\ell-1)^{n_\ell}(\ell-2)^{n_{\ell-1}} \ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

or

$$((\ell-1) \ldots -(\ell-1))((\ell-1)^{n_\ell}(\ell-2)^{n_{\ell-1}} \ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

2. Denote the cuspidal string

$$(\mathscr{a}, \mathscr{a}-1, \ldots \mathscr{b})((\ell+m)(\ell+m-1) \ldots ((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$$

and set $m = 0$.

— If $\mathscr{b} = 0, \mathscr{a} \geq (\ell+1)$; one inserts elements from the left hand segment to the right hand segment to obtain :

$$(0)(\mathscr{a}^{n_\ell}(\mathscr{a}-1)^{n_{\ell-1}} \ldots (\ell-i)^{n_{\ell-i}+1} \ldots 1^{n_1+1}0^{n_0})$$

— If $\mathscr{b} \geq 1$ and $\mathscr{b} \leq \mathscr{a} \leq \ell$, one withdraws the zeroes and subsequent higher numbers to get :

$$(\ell(\ell-1)\ldots-\ell)((\ell-1)^{n_\ell}(l-2)^{n_{\ell-1}}\ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

or

$$((\ell-1)\ldots-(\ell-1))((\ell-1)^{n_\ell}(\ell-2)^{n_{\ell-1}}\ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

— Again to get minimal Langlands parameters one needs to choose a left hand segment $(a, a-1, \ldots b)$ so that there is an obstruction to *add* elements in the right hand residual segment : i.e. $a \geq \ell$; and there is an obstruction to *withdraw* elements from the right hand residual segment : i.e. $-\ell \geq b \geq 0$. Therefore for $0 \geq b \geq -\ell$ and $\ell \geq a \geq b$, we get *unalterable* cuspidal string.

□

Lemma 107. *Denote E the set of indices $i \in \{2, \ldots, l-1\}$ such that $n_{i-1} = n_i$. Assume $E \neq \emptyset$. The cuspidal string given by $(a, a-1, \ldots b)(\underline{n})$ is unalterable for all segments $(a, a-1, \ldots b)$ under the following conditions :*

— *When $n_0 = \frac{n_1-1}{2}, -\ell \leq b \leq -1$ and $b \leq a \leq \ell$, or $b = 1$ and $(\ell+1) > a > b$. Or for $j \in E$, if $b = j$ or $-j$, and $a > b$. In particular,*

$$(j)(\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

is unalterable.

— *When $n_0 = \frac{n_1}{2}; l \leq b \leq 0$ and $b \leq a \leq l$. Or for $j \in E$, if $b = j$ or $-j$, and $a > b$. In particular,*

$$(j)(\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

is unalterable.

— *When $n_2 = n_1 - 1$ and $n_0 = \frac{n_1}{2}$, if $b = a = 1$.*

The exceptions take one of the three following forms : $(a \ldots 0 \ldots -(j-1))$ with $a \neq (j-1)$; $(j-1 \ldots 0 \ldots b)$ with $b \neq j'-1$; or $(j-1, \ldots, j'-1)$ if $j, j' \in E$.

Remark 15. The case of $E = \emptyset$ is Lemma 106.

Proof. The same case by case study detailed in the proof of Lemma 106 can be repeated in this context to prove that if $n_0 = 1/2(n_1-1), b \geq 1$, and/or $a \geq \ell$; or $n_0 = \frac{n_1}{2}$, with $0 \leq b$ and/or $a > l$ the cuspidal string is unalterable.

However, let's assume $n_0 = \frac{n_1-1}{2}$, with $-\ell \leq b \leq -1$ and $b \leq a \leq \ell$, or $b = 1$ and $(\ell+1) > a > b$ or $n_0 = \frac{n_1}{2}$ with $\ell \leq b \leq 0$ and $b \leq a \leq \ell$.

In

$$(a, a-1, \ldots b)((\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

set $m = 0$.

Suppose the indices j, j' belong to E, that is $n_{j-1} = n_j, n_{j'-1} = n_{j'}$.

Then the cuspidal string

$$(a, a-1, \dots b)((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

with $a \neq (j-1)$ can be transformed to

$$(a(a-1)\dots 0, \dots -a)((\ell-1)^{n_\ell}(\ell-2)^{n_{\ell-1}}\dots 2^{n'_2}1^{n'_1}0^{n'_0})$$

the cuspidal string

$$((j-1)\dots b)(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

with $b \neq j'-1$ can be transformed by insertion from the left hand segment to the right hand segment to

$$(j-2, \dots 0)(\ell^{n'_\ell}(\ell-1)^{n'_{\ell-1}}(\ell-2)^{n'_{\ell-2}}\dots 2^{n'_2}1^{n'_1}0^{n'_0})$$

or

$$(j-2, \dots -(j-2))(\ell^{n'_\ell}(\ell-1)^{n'_{\ell-1}}(\ell-2)^{n'_{\ell-2}}\dots 2^{n'_2}1^{n'_1}0^{n'_0})$$

where $n'_i = n_i + 1$ for i in

$$\{j, j-1\}\,; or (0)(\ell^{n'_\ell}(\ell-1)^{n'_{\ell-1}}(\ell-2)^{n'_{\ell-2}}\dots 2^{n'_2}1^{n'_1}0^{n'_0})$$

for all indices i in $\{1, \dots, j\}$.

$$(j-1\dots -(j'-1))(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

can be transformed to a residual segment, or altered by insertion or removal from the right hand residual segment.

$$(a, a-1, \dots b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

More generally if there are n such indices j with $n_{j-1} = n_j$, we get unalterable cuspidal string except for the $2n$ different cuspidal strings

$$(a, a-1, \dots j-1)(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

with $a \neq (j-1)$ or

$$((j-1)\dots b)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

with $\mathscr{b} \neq j' - 1$; and the $(n)(n-1)$ different cuspidal strings of the form :

$$(j-1, \ldots - (j'-1))\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$$

for j, j' in E. $\square$

Eventually, we characterize cuspidal strings giving minimal Langlands parameter denoted ν_{min}^+ for the order given in Lemma 31 on Langlands parameters obtained with Langlands data $(P, \sigma|.|^s \otimes \tau, \nu)$ where P is a maximal parabolic subgroup.

Proposition 108. *Let τ' be an essentially square-integrable representation of the Levi subgroup M' of a maximal parabolic subgroup P' of G. If the cuspidal string associated to the representation τ' is unalterable as characterized in Lemmas 106 and 107, the corresponding Langlands parameter ${\nu'_{min}}^+$ is minimal for the order given in Lemma 31 on Langlands parameters obtained with Langlands data (P, τ, ν) where P is a maximal parabolic subgroup.*

Proposition 109. *Let τ be a generic discrete series representation of M_s the Levi subgroup of a parabolic subgroup P_s of G. The cuspidal string associated to the representation τ is of the form*

$$(\gamma, \ldots, \delta)(\iota, \ldots, \kappa)(l')^{n'_l}(l'-1)^{n_{l'-1}} \ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

with $\gamma \geq \delta$, and $\iota \geq \kappa$,and is necessarily obtained from an unalterable cuspidal string of the form

$$(\mathscr{a}, \ldots, \mathscr{b})((\ell+m) \ldots (\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$$

as characterized in Proposition 108. The latter unalterable cuspidal string has an associated Langlands parameter ν_{min}^+.

Consider $\nu_{min} = (s_1, s_2)$ with $s_1 \geq s_2 \geq 0$ and $\nu_{min} < \nu_{min}^+$ for the order given in Lemma 31 on Langlands parameters. The Langlands parameter ν_{min} is minimal for this order. That is the standard module of the form $I_{P_s}^G(\tau, \nu_{min})$ is irreducible.

Let E be the set defined in Lemma 107. Assume the unalterable cuspidal string is of the form

$$(\mathscr{a}, \ldots \mathscr{b})((\ell+m)(\ell+m-1) \ldots (\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$$

with $\mathscr{b} = j$ or $-j$. Then $s_2 = 0$ or $s_2 = 1/2$ and $s_1 = j$ or $s_1 = 1$.

Else, i.e. if the unalterable cuspidal string takes any other form given in Lemmas 106 and 107, the minimal Langlands parameter writes $\nu_{min} = (s_1, s_2)$ with $s_1 = s_2 = 0$ or $s_1 = 1 \geq s_2 = 0$ and is obtained by splitting the first segment of the unalterable cuspidal string.

Proof. If $\nu_{min}^+ = 0$ then $\nu_{min} = \nu_{min}^+$ and the module $I_{P_{min}}^G(\tau_{min}, \nu_{min})$, with P_{min} a maximal parabolic subgroup, is irreducible.

Otherwise, assume $\nu_{min}^{+} \geq 0$. We explicit below the form of the cuspidal string giving Langlands parameter ν_{min} smaller than ν_{min}^{+} for the order given in Lemma 31.

First, consider an unalterable cuspidal string $(a, \dots b)(\underline{n})$, let j be in E as defined in Lemma 107 with $b = j$ or $-j$, and any $a > b$. If $m = 0$, then $(\ell + m)$ is just ℓ.

If $a \leq (\ell + m)$, it is transformed in

$$((k, \dots, -k)(j)((\ell')^{n'_\ell}(\ell' - 1)^{n_{\ell'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

if $n_0 \leq 2$ and

$$((k, \dots, -k)(j(j-1) \dots 0 \dots -(j-2))((\ell')^{n'_\ell}(\ell' - 1)^{n_{\ell'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

if $n_0 \geq 2$. If $a \geq (\ell + m)$, it is transformed in

$$((k+1), k, \dots, -k)(j)((\ell')^{n'_\ell}(\ell' - 1)^{n_{\ell'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

or respectively

$$((k, \dots, -k)(j(j-1) \dots 0 \dots -(j-2))((\ell')^{n'_\ell}(\ell' - 1)^{n_{\ell'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

Clearly, one observes that for these cases : $s_2 = 0$ or $s_2 = 1/2$ and $s_1 = j$ or $s_1 = 1$.

Let n_0 be either $\frac{n_1}{2}$ or $\frac{n_1 - 1}{2}$, $n_i = n_{i-1} + 1$ for all i but a set of elements E, possibly non empty, , let $-l \leq b \leq 0$ and $b < a \leq l$ such that

$$(a, a-1, \dots b)((\ell + m)(\ell + m - 1) \dots ((\ell + 1)\ell^{n_\ell}(\ell - 1)^{n_{\ell-1}}(\ell - 2)^{n_{\ell-2}} \dots 2^{n_2} 1^{n_1} 0^{n_0})$$

is the unalterable cuspidal string given in Lemmas 106 and 107.
Then by removing elements from the right hand residual segment, the cuspidal string can be transformed into

$$(a, \dots, 0, \dots - a)(k \dots 0 \dots - k)((l')^{n'_l}(l' - 1)^{n_{l'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

In particular, if $n_0 = \frac{n_1}{2}$ and $b = 0$, then $k = 0$, and the transformed cuspidal string is just :

$$(a, \dots, 0, \dots - a)(0)((l')^{n'_l}(l' - 1)^{n_{l'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

Then $\nu_{min} = (s_1, s_2) = (0, 0)$.

If $b = 1$, that is from Lemmas 106 and 107 , $n_0 = \frac{n_1 - 1}{2}$; the cuspidal string is transformed into

$$(a, \dots, 0, \dots - a)(1)((\ell')^{n'_\ell}(\ell' - 1)^{n_{\ell'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

where $n'_0 = \frac{n'_1 - 1}{2}$.

Indeed, one needs to remove a zero to obtain the segment $(a, \dots, 0, \dots - a)$, and under the constraint $n_0 = \frac{n_1 - 1}{2}$, one needs therefore to remove two ones. Therefore, there is necessary a one remaining. In this case, $\nu_{min} = (s_1, s_2) = (1, 0)$.

In case there are different cuspidal strings of the forms

$$(a, \dots, 0, \dots - a)(k \dots 0 \dots - k)((\ell')^{n'_\ell}(\ell' - 1)^{n_{\ell'-1}} \dots 2^{n'_2} 1^{n'_1} 0^{n'_0})$$

(i.e. k may take different values) leading to the same values of $s_1 \geq s_2 = 0$, one chooses the minimal length segment on the cuspidal string to pin down the triple $(\pi_1|.|^{s_1} \otimes \pi_2|.|^{s_2} \otimes \tau_{min}, P_s, \nu_{min})$.

The only exceptions to this « splitting » procedure occur when $n_0 = 1, n_1 = 2$ or $n_1 = 3$. If $b = 1$, the cuspidal string

$$(a, a-1, \dots b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

cannot be splitted, and gives $\nu_{min}^{+} = \nu_{min} > 0$.

□

Proposition 110. *Let τ be a generic discrete series of M, a maximal Levi subgroup of G (of type B, C or D), s a strictly positive complex number. Its cuspidal string takes the form : $(a, a-1, \dots b)(\underline{n})$.*

The cuspidal string can always be reorganized to obtain a minimal Langlands parameter with respect to the order given in Lemma 31.

That is for a given cuspidal string λ associated to a Langlands' data (P, τ_ν) with P a maximal parabolic subgroup and τ as above, we can explicit the form of the Langlands parameter ν_{min} such that $I_{P_{min}}^G(\tau_{min_{\nu_{min}}})$ is the irreducible generic subquotient of $I_P^G(\tau_\nu)$.

Proof. Consider the following cuspidal string :

$$(a, a-1, \dots, b)((\ell+m)(\ell+m-1)\dots(\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

Either this cuspidal string is unalterable as characterized in Lemmas 106 and 107 ; either it is of the following forms :

- First, assume $n \geq m \geq z \geq 1$, and $a = \ell + n, b = -(\ell + z)$. That is the cuspidal string takes the form :

  $$(\ell+n, \dots, 0, \dots -(\ell+z))((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

  one inserts all elements from the left hand segment to the right hand residual segment to get a residual point.

  $$((\ell+n)(\ell+n-1)\dots(\ell^{n'_\ell}(\ell-1)^{n'_{\ell-1}}\dots 2^{n'_2}1^{n'_1}0^{n'_0})$$

- $m \geq n \geq z \geq 1$ is treated similarly.
- Set $n_0 = \frac{n_1-1}{2}$, $b = 1$. For any m -if $m = 0$, $\ell + m = \ell$- if $b = 1 \leq a = (\ell + n)$

one transforms the cuspidal string to

$$((\ell+n)\ldots 0\ldots-(\ell+m))(\ell'^{n_{\ell'}}(\ell'-1)^{n_{\ell'-1}}(\ell'-2)^{n_{\ell'-2}}\ldots 2^{n'_2}1^{n'_1}0^{n'_0})$$

If $m=0, n\geq 1$, the cuspidal string

$$((\ell+n),\ldots,\ell,\ldots,1)(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

transforms to

$$(a',\ldots,1)((\ell+n),\ldots,\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

where $a'\leq(\ell+n)$, and this is an unalterable cuspidal string as characterized in Lemma 106.

If the cuspidal string can be transformed to a residual segment, then there is no Langlands parameter ν_{min} to be characterized, namely the unique irreducible generic subquotient is a discrete series. From an unalterable cuspidal string, one applies first the procedure of Lemmas 106 and 107 to characterize ν^+_{min}. Then, using the procedure given in the proof of Proposition 109, one characterizes ν_{min}.

□

G.2. C_l

Lemma 111. *Consider the cuspidal string given by*

$$(a,\ldots,b)((\ell+m)/2,(\ell+m-1)/2,\ldots((\ell+1)/2,\ell/2^{n_{\ell/2}},(\ell-2)/2^{n_{(\ell-2)/2}},(\ell-4)/2^{n_{(\ell-4)/2}},\ldots 3^{n_{3/2}},1/2^{n_{1/2}})$$

with $n_{i-1}=n_i+1; n_{1/2}=n_{3/2}+1$, $n_{\ell/2}=1, n_{\ell-1/2}=2$, and (half)-integers $b<a$. This cuspidal string is unalterable if and only if : $b=1/2$ or $b=-1/2$ and $a\leq\ell/2$

Proof. Consider first the case $m>2$. If $a=(\ell+n)/2<(\ell+m)/2$ and $a\geq(\ell+2)/2$, for any b such that $-\ell/2\leq b$ then one gets the residual segment

$$(\ell+m)/2,(\ell+m-2)/2,\ldots((\ell+n+2)/2^{n_{\ell/2}},(\ell+n)/2^{n_{\ell-2}},\ldots,b^{n_b+1},\ldots,1/2^{n_{/2}})$$

If $n>m$ one interchanges n and m in the previous sequence, i.e,

$$(\ell+n)/2,\ldots,(\ell+n-2)/2\ldots,(\ell+m+2)/2^{n_{\ell/2}},(\ell+m)/2^{n_{(\ell-2)/2}},\ldots,1/2^{n_{1/2}})$$

If $a=(\ell+n)/2=(\ell+m)/2$, the unalterability of the cuspidal string depends on the value of b. In this case, the results are the same that if we assume $m=0$.

So, let $m=0$.

— If $a \geq \ell/2$, and $b \geq 1/2$, one gets a residual segment.

Now, assume $a \leq \ell/2$:

— If $b > 1/2$, then one withdraw elements on the right to get

$$(a, \ldots, 1/2)(\ell/2^{n_{\ell/2}}, (\ell-2)/2^{n_{(\ell-2)/2}}, (\ell-4)/2^{n_{\ell-4/2}}, \ldots$$
$$\ldots, (b-2)/2^{n_{(b-2)/2}-1}, \ldots, 3/2^{n_{3/2}-1}, 1/2^{n_{1/2}-1})$$

— If $b = 1/2$ the cuspidal string

$$(a, \ldots, 1/2)(\ell/2^{n_{\ell/2}}, (\ell-2)/2^{n_{(\ell-2)/2}}, (\ell-4)/2^{n_{\ell-4/2}}, \ldots, 3/2^{n_{3/2}}, 1/2^{n_{1/2}})$$

is unalterable.

— If $b = -1/2$, we similarly get an unalterable cuspidal string.

— If $b \leq 0$, one withdraws elements on the right hand residual segment to get :

$$(a, \ldots, -a)(\ell/2^{n'_{\ell/2}}, (\ell-2)/2^{n'_{(\ell-2)/2}}, (\ell-4)/2^{n'_{(\ell-4)/2}}, \ldots, 3/2^{n'_{3/2}}, 1/2^{n'_{1/2}})$$

with $n'_i = n_i - 1$ for $i \in \{b, \ldots, a\}$.

□

Lemma 112. *Denote E the set of indices $i \in \{3/2, \ldots, (\ell-2)/2\}$ such that $n_{i-1} = n_i$. Assume $E \neq \emptyset$, and let $j/2$ be an element in E, for an odd integer j. The cuspidal string given by*

$$(a, a-1, \ldots, b)((\ell+m)/2, (\ell+m-2)/2, \ldots$$
$$\ldots, ((\ell+2)/2, \ell/2^{n_{\ell/2}}, (\ell-2)/2^{n_{(\ell-2)/2}}, (\ell-4)/2^{n_{(\ell-4)/2}}, \ldots, 3/2^{n_{3/2}}, 1/2^{n_{1/2}})$$

is unalterable for all residual segments $(a, a-1, \ldots, b)$ under the following conditions : $a \leq \ell$ and $b = -1/2$, $b = 1/2$, or $b = j/2$ up to certain exceptions.

The set of exceptions takes the form : $(a, \ldots, 0, \ldots, -(j-1)/2)$. In the more specific case of $n_i = 2$ for all i except $\ell/2$, and $n_{\ell/2} = 1$, all residual segments with $b \leq 0$ and $a \leq \ell$ are alterable.

Proof. The arguments given in the previous lemma 111 follow.

If $b = j/2$, since we cannot withdraw $(j-2)/2$ from the right hand residual segment (else we would have $n'_{(j-2)/2} = n'_{j/2} - 1$), the cuspidal string $(a, \ldots, j/2)$ is unalterable.

$$(a, \ldots, 0, \ldots, -(j-2)/2)((\ell+m)/2, (\ell+m-2)/2, \ldots$$
$$\ldots, ((\ell+2)/2, \ell/2^{n_{\ell/2}}, (\ell-2)/2^{n_{(\ell-2)/2}}, (\ell-4)^{n_{(\ell-4)/2}}, \ldots, 3/2^{n_{3/2}}, 1/2^{n_{1/2}})$$

is tranformed into

$$(a,\dots,0,\dots,-a)((\ell-2)^{n_\ell}(\ell-4)^{n_{\ell-2}}\dots j/2^{n_{j/2}-1}(j-2)/2^{n_{(j-2)/2}}\dots 1/2^{n_{1/2}})$$

In the case of $n_i = 2$ for all i except $\ell/2$, and $n_{\ell/2} = 1$,

$$(a,\dots,b)(\ell/2,(\ell-2)/2^2,(\ell-4)/2^2,\dots,3/2^2,1/2^2)$$

If $b \leq 0$ and $a \leq \ell/2$, the cuspidal strings transforms into

$$(a,\dots,-a)(\ell'/2,(\ell'-2)/2^{n_{(\ell'-2)/2}},\dots,(b+1)^{n_{b+1}-1},b^{n_b},\dots,1/2^{n_{1/2}})$$

□

Proposition 113. *Let τ be a generic discrete series of M, a maximal Levi subgroup of G, s a strictly positive complex number. Its cuspidal string takes the form :*

$$(a,a-1,\dots b)((\ell+m)/2,(\ell+m-2)/2,\dots,(\ell+2)/2,\ell/2^{n_{\ell/2}},(\ell-2)/2^{n_{(\ell-2)/2}}(\ell-4)^{n_{(\ell-4)/2}}\dots 3/2^{n_{3/2}}1/2^{n_1}$$

The cuspidal string can always be reorganized to obtain a minimal Langlands parameter with respect to the order given in Lemma 31.

That is for a given cuspidal string λ associated to a Langlands' data (P,τ,ν) with P a maximal parabolic subgroup and τ as above, we can explicit the form of the Langlands parameter ν_{min} such that $I_{P_{min}}^G(\tau_{min_{\nu_{min}}})$ is the irreducible generic subquotient of $I_P^G(\tau_\nu)$.

Proof. Similar analysis that for the case B_l. Details are left to the reader. □

G.3. D_l

Residual points of type D_l are of the following form :

$$(a,a-1,\dots b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

1. $n_i = 1$ for all $i \geq \ell$ and $n_0 = 1$, $n_i = 2$ for all $i \in \{2,\dots,\ell-1\}$.
2. $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 \geq 2$, $n_0 = \begin{cases} \frac{n_1}{2} & \text{if } n_1 \text{ is even} \\ \frac{n_1+1}{2} & \text{if } n_1 \text{ is odd} \end{cases}$

Lemma 114 (type 1)**.** *If the cuspidal string has the form*

$$(a,\dots,b)(\ell(\ell-1)(\ell-2)\dots 0)$$

i.e. $n_i = 1$ for all i, and $\ell \geq a \geq b \geq 1$ the cuspidal string is unalterable.

If $n_i \geq 2$ for some i and any $\ell \geq a \geq b > 1$ the cuspidal string is unalterable.

Proof. Assume $n_i = 1$ for all i. First, if $a \geq \ell$, one transforms

$$(a, a-1, \dots b)(\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

to

$$(a', \dots, b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

by importing all elements from a to a' from the left hand segment to the right hand residual segment.

If $\ell \geq a, b \leq 0$, one withdraws elements from the right hand residual segment to obtain $(a, \dots, -a)(\underline{n})$

If $b > 1$, the cuspidal string is unalterable since one can neither add (resp. withdraw) elements from (resp to) the right hand residual segment. Indeed, notice that if one were to add a zero on the right hand residual segment, one would need to add at least four one.

Assume now that $n_i \geq 2$ for some indices i.

Then $(a, \dots, 1)(\ell(\ell{-}1)\dots 10)$ transforms to a residue point : $(\ell \dots a^{n_a+1}\dots 2^{n_2+1}1^{n_1+1}0)$. If $b = 0, (a, \dots, 0)(\ell(\ell-1)\dots 10)$ transforms to $(0)(\ell \dots a^{n_a+1}\dots 2^{n_2+1}1^{n_1+1}0)$

If $b \leq -1$, then $(a, \dots, b)(\ell(\ell-1)\dots 10)$ transforms to $(\ell \dots - a)(b \dots 10)$. □

Lemma 115 (type 2)**.** *Consider the cuspidal string given by*

$$(a, \dots, b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1})$$

with $n_{i-1} = n_i + 1; n_1 = n_2 + 1$*, and* $b < a$*.*

This cuspidal string is unalterable if and only if :

$a = (\ell + m) = (\ell + n)$.

If $n_0 = \frac{n_1}{2} - 1$,

$$(1)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

is unalterable.

Let $n_0 = 2$ *and* $a \leq \ell$*. If* $n_0 = \frac{n_1+1}{2} - 1$*, any* b *; If* $n_0 = \frac{n_1}{2} - 1$*, any* $b \leq -1$

When $n_{i-1} = n_i + 1$ *for all* i *and* $n_0 = \frac{n_1+1}{2} - 1, a \leq \ell,\ b \leq -1$

When $n_{i-1} = n_i + 1$ *for all* i *and* $n_0 = \frac{n_1}{2} - 1$ *;* $a \leq \ell$ *;* $b = 1$ *or* $b \leq -1$

Proof. First assume that $m \geq 1$, and $n \geq m$, then either we can insert all elements to get a residual segment, or we can insert only partly (for instance if $b \leq -1$, we will be left with $(b, \dots, -b)(\ell'(\ell'-1)^{n_{\ell'-1}}\dots 1^{n_1'}0^{n_0-1})$.

As in the previous proofs, if $1 \leq n \leq m$, we interchange n and m and treat this configuration as the previous one.

Now assuming $m=0$. If $a \geq \ell+1$, for any b, either we insert all elements in the right hand residual segment and get a residual segment or we insert only certain elements and get we will be left with $(b,\dots,-b)(\ell'(\ell'-1)^{n_{\ell'-1}}\dots 1^{n'_1}0^{n_0-1})$.

Let $a \leq \ell$.

Assume first that $n_0=2$.

- If $n_0 = \frac{n_1}{2}-1$, and $b>1$, we remove elements in $\{1,b\}$ on the right hand residual segment, and the cuspidal string becomes $(a,\dots,1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b(b-1)^{n_{b-1}-1}\dots 1^{n_1-1}}0^{n_0}$.
- If $n_0=\frac{n_1}{2}-1$, and $b=0$, one removes elements from the right hand residual segments to build $(a,\dots,-a)(\ell'(\ell'-1)^{n_{\ell'-1}}\dots 1^{n'_1}0^{n_0})$.
- Else, if $n_0=\frac{n_1+1}{2}-1$, one cannot withdraw a one, so for any $b\geq 0$ the cuspidal string is unalterable.
- for $b\leq -1$, $n_0=\frac{n_1}{2}-1$ or $n_0=\frac{n_1+1}{2}-1$, the cuspidal string is unalterable.

Assume $n_0 \geq 3$.

- If $b>1$, can withdraw a zero and two ones, and subsequent higher numbers to obtain $(a,\dots,-a)(\ell'(\ell'-1)^{n_{\ell'-1}}\dots 1^{n_1-2}0^{n_0-1})$.
- If $b\leq -1$ it is unalterable.
- Let $b=1$.
  If $n_0=\frac{n_1+1}{2}-1$, one withdraws a zero and a one on the right and subsequent higher numbers to get $(a,\dots,-a)(\ell'(\ell'-1)^{n_{\ell'-1}}\dots 1^{n'_1}0^{n_0})$. Else if $n_0=\frac{n_1}{2}-1$, it is unalterable.

□

Lemma 116. *Denote E the set of indices $i\in\{2,\dots,\ell-1\}$ such that $n_{i-1}=n_i$. Assume $E\neq\emptyset$, and let j be an element in E. The cuspidal string given by*

$$(a,a-1,\dots,b)((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1})$$

is unalterable for all residual segments $(a,a-1,\dots,b)$ under the following conditions : $a\leq\ell$ and $b\leq -1$ or $b=j$ up to certain exceptions.

The set of exceptions take the form : $(j-1,\dots,b)$ or $(a,\dots,0,\dots,-(j-1))$.

In the more specific case of $n_i=2$ for all i except ℓ, and $n_\ell=1$, all segments with $b\leq 0$ and $a\leq\ell$ are alterable.

Proof. The arguments given in the previous Lemmas 114 and 115 follow. Let $j\in E$. We discuss according the different forms of the segment $(a,\dots,b)$:

- if $a=j-1$, one insert all elements from the left hand segment to the right hand residual segment, and obtain a residue point.
- for $\ell+m\geq a$ and $b=-(j-1)$, one remove elements from the right hand residual segment (j to ℓ) to obtain $(a,\dots,(-\ell-1))$.

In the more specific case of $n_i = 2$ for all i except ℓ, and $n_\ell = 1$, all segments with $\mathscr{b} \leq 0$ and $\mathscr{a} \leq \ell$ are alterable. Indeed, possibly we remove elements from the right hand residual segment to get on the left $\mathscr{b}' < \mathscr{b}$ and $\mathscr{a}' > \mathscr{a}$. $\square$

Proposition 117. *Let τ be a generic discrete series of M, a maximal Levi subgroup of G, s a strictly positive complex number. Its cuspidal string takes the form :*

$$(\mathscr{a}, \mathscr{a}-1, \dots \mathscr{b})((\ell+m)(\ell+m-1)\dots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 2^{n_2}1^{n_1}0^{n_0})$$

The cuspidal string can always be reorganized to obtain a minimal Langlands parameter with respect to the order given in Lemma 31.

That is for a given cuspidal string λ associated to a Langlands' data (P, τ, ν) with P a maximal parabolic subgroup and τ as above, we can explicit the form of the Langlands parameter ν_{min} such that $I_{P_{min}}^G(\tau_{min}, \nu_{min})$ is the irreducible generic subquotient of $I_P^G(\tau, \nu)$.

Proof. repeat more or less the proof for B_l, leave the details to the reader. $\square$

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2. Symplectic Models for Unitary Groups

In analogy with the study of representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$, where F is a local field, in this paper we study representations of $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. (Only quasi-split unitary groups are considered in this paper since they are the only ones which contain $\mathrm{Sp}_{2n}(F)$.) We prove that there are no cuspidal representations of $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ for F a non-archimedean local field. We also prove the corresponding global theorem that there are no cuspidal representations of $\mathrm{U}_{2n}(\mathbb{A}_k)$ with nonzero period integral on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$ for k any number field or a function field. We completely classify representations of quasi-split unitary group in four variables over local and global fields with nontrivial symplectic periods using methods of theta correspondence. We propose a conjectural answer for the classification of all representations of a quasi-split unitary group distinguished by $\mathrm{Sp}_{2n}(F)$.

1. Introduction

Among the many examples studied about automorphic representations of $G(\mathbb{A})$ which have nonzero period integrals (where $\mathbb{A}$ is the adele ring of a number field k) :

$$\int_{H(k)\backslash H(\mathbb{A})} f(h)dh \not\equiv 0,$$

for $f \in \Pi$, an automorphic representation of $G(\mathbb{A})$, for G a reductive algebraic group over the number field k, and H an algebraic subgroup of G defined over k, one of the most complete and beautiful works is due to O. Offen and E. Sayag about symplectic periods of automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$ (for $H(\mathbb{A}) = \mathrm{Sp}_{2n}(\mathbb{A})$), cf. [OS1] and [OS2] for both local and global results for the pair $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$.

One of the early results on symplectic periods is due to Heumos and Rallis who proved that there are no cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A})$ with nonzero symplectic period since in fact there are no generic representations of $\mathrm{GL}_{2n}(k_v)$ which are distinguished by $\mathrm{Sp}_{2n}(k_v)$. (For a subgroup H of a group G, a representation π of G is said to be distinguished by H if there exists a nonzero linear form $\ell : \pi \to \mathbb{C}$ such that $\ell(hv) = \ell(v)$ for all $h \in H$, and $v \in \pi$.)

In analogy with works on symplectic periods of automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$, one can consider similar questions by replacing $G = \mathrm{GL}_{2n}$ by $G = \mathrm{U}_{2n}$, a unitary group defined by a hermitian form on a $2n$-dimensional vector space V over K, where K is a quadratic extension of k.

Observe that

$$\mathrm{Sp}_{2n}(k_v) \subset \mathrm{U}_{2n}(k_v),$$

when one takes unitary group in $2n$-variables which is quasi-split. For example, let

$$A = \begin{pmatrix} & & & & & & i \\ & & & & & i & \\ & & & & i & & \\ & & & * & & & \\ & & -i & & & & \\ & -i & & & & & \\ -i & & & & & & \end{pmatrix}$$

where $i \in K_v^\times$ with $\bar{i} = -i$. The matrix A is hermitian, but iA is symplectic, and therefore the unitary group defined by A contains the symplectic group defined by iA.

Since we now have $\mathrm{Sp}_{2n}(k_v) \subset \mathrm{U}_{2n}(k_v)$, it is a meaningful question to consider representations on $\mathrm{U}_{2n}(k_v)$ which are distinguished by $\mathrm{Sp}_{2n}(k_v)$, or automorphic representations of $\mathrm{U}_{2n}(\mathbb{A})$ which have nonzero period integral on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})$. In fact this question is already considered by Lei Zhang who proved, cf. Theorem 1.1 in [Zh1] that $(\mathrm{U}_{2n}(k_v), \mathrm{Sp}_{2n}(k_v))$ is a Gelfand pair, i.e., the space of $\mathrm{Sp}_{2n}(k_v)$-invariant linear forms on any irreducible admissible representation of $\mathrm{U}_{2n}(k_v)$ is at most one dimensional, for k_v any local field. In [Zh2] he further proved that there are no tame supercuspidal representations of $\mathrm{U}_{2n}(k_v)$ distinguished by $\mathrm{Sp}_{2n}(k_v)$.

In this work, we prove that there are no cuspidal representations of $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ for F a non-archimedean local field — thus completing the work of Lei Zhang. We also prove the corresponding global theorem that there are no cuspidal representations of $\mathrm{U}_{2n}(\mathbb{A}_k)$ with nonzero period integral on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$ for k any number field or a function field.

Our proof of non-existence of cuspidal representations $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ works as well to prove that there are no cuspidal representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$, thus giving another proof of the theorem of Heumos and Rallis, and in fact has consequences for representations of $\mathrm{SL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ about which we make a general conjecture and prove it in some cases. We also propose a conjectural answer for the classification of all representations of a quasi-split unitary group with symplectic period.

We completely classify representations of quasi-split unitary group in four variables over local and global fields with nontrivial symplectic periods using methods of theta correspondence.

Our analysis with theta correspondence uses relationship of $\mathrm{U}_4(F)$ with a certain orthogonal group in 6 variables, and symplectic group $\mathrm{Sp}_4(F)$ with a certain orthogonal group in 5 variables; especially the first identification seems not so standard, so we have taken some pains to elaborate on these.

In a totally independent and almost simultaneous work [OM], O. Offen and A. Mitra following [AGRS] have also proved that the pair (U(2n), Sp(2n)) is a vanishing pair in the sense of [AGRS]; our work — using a very different approach than [OM] — has more consequences for representations of U(2n) distinguished by Sp(2n), and a formulation of a general conjecture in this regard.

2. Notation

We will use F to denote either a general field, or a local field, and k will be used to denote a number field. If F is a local field, it will always come equipped with a fixed non-trivial additive character $\psi : F \to \mathbb{C}^\times$. For a number field k, we will let $\mathbb{A} = \mathbb{A}_k$ denote its adele ring, and we will always fix a non-trivial additive character $\psi_0 : \mathbb{A}_k/k \to \mathbb{C}^\times$.

Given a vector space V over a field F, we will let $V^\vee$ denote the dual vector space over F. If F is a local field with a fixed non-trivial character $\psi : F \to \mathbb{C}^\times$, observe that the dual vector space $V^\vee$ can also be identified to the set of all characters $\widehat{V}$ of V (the Pontryagin dual) :

$$\ell \in V^\vee \longrightarrow \hat{\ell} \in \widehat{V},$$

defined by

$$\hat{\ell}(v) = \psi(\ell(v)).$$

For example, for a symplectic vector space $W = X + X^\vee$, with X and $X^\vee$ maximal isotropic subspaces of W, let P_X be the Siegel parabolic in $\mathrm{Sp}(W)$ stabilizing X with unipotent radical N_X which is the vector space of symmetric elements $\{\phi \in \mathrm{Hom}\,(X^\vee, X) | \phi = \phi^\vee\} \cong \mathrm{Sym}^2 X$. If we denote the set of symmetric elements of $\mathrm{Hom}\,(X^\vee, X)$ by $\mathrm{SHom}(X^\vee, X)$, then the natural non-degenerate pairing :

$$\begin{tikzcd} \mathrm{Hom}\,(X^\vee, X) \times \mathrm{Hom}\,(X, X^\vee) \arrow[r] \arrow[rr, bend left, phantom, ""] & \mathrm{Hom}\,(X^\vee, X^\vee) \arrow[r, "\mathrm{tr}"'] & F, \end{tikzcd}$$

gives rise to a non-degenerate pairing :

$$\mathrm{SHom}(X^\vee, X) \times \mathrm{SHom}(X, X^\vee) \longrightarrow F,$$

identifying the dual of $\mathrm{SHom}(X^\vee, X)$ to $\mathrm{SHom}(X, X^\vee)$, and therefore, the character group of $\mathrm{SHom}(X^\vee, X)$ is identified to $\mathrm{SHom}(X, X^\vee)$ (the identification of course depends on the choice of the non-trivial character $\psi : F \to \mathbb{C}^\times$ which will be fixed throughout the paper).

If (V, q) is a quadratic space over a field F, $\mathrm{O}(V)$ denotes the associated orthogonal group over F. We will use the notation $\mathrm{O}(m, n)$, which is usually used in the context of real groups, to denote any orthogonal group whose rank over F is $\min\{m, n\}$; the notation $\mathrm{O}(m, n)$ does not give full information on the quadratic form, or the isomorphism class of the group, but still carries very useful information specially when dealing with orthogonal groups which are split or quasi-split, i.e., $\mathrm{O}(m, n)$ with $|m - n| \leq 2$. If the orthogonal group is $\mathrm{O}(m, m+2)$, then it is a quasi-split group over F, split by a unique quadratic field extension of F ; for us this quadratic extension will always be E, the quadratic extension of F involved in defining the hermitian form underlying our unitary groups.

We will similarly denote unitary groups by $\mathrm{U}(m, n)$ to be any unitary group whose F rank is $\min\{m, n\}$. We will use the notation $\mathrm{O}(m), \mathrm{U}(m)$ to denote any orthogonal or unitary group defined by a quadratic or hermition space of dimension m, or $\mathrm{O}_m(F)$, $\mathrm{U}_m(F)$ if we want to be explicit about F.

Given a vector space V over F together with a quadratic form $q : V \to F$, and $a \in F^\times$, we will abbreviate $a \cdot V$ to be the quadratic space with V as the underlying vector space, and $a \cdot q$ as the quadratic form on V. Note that although $\mathrm{O}(a \cdot V) = \mathrm{O}(V)$, for much of the considerations in this paper which deal with theta correspondence, it will be important to treat $a \cdot V$ as a different quadratic space from V with $a \cdot V$ isomorphic to V if and only if there is an automorphism g of V such that $q(gv) = a \cdot q(v)$ for all $v \in V$. For example, if E is a separable quadratic extension of a field F, then E considered as a two dimensional vector space over F carries the quadratic form $q = \mathbb{N}m$ where $\mathbb{N}m(e) = e\bar{e}$. Then for $a \in F^\times$, the quadratic space $a \cdot E$ is isomorphic to E if and only if $a \in \mathbb{N}m(E^\times)$.

3. Clifford theory à la Bernstein-Zelevinsky

This section written for the purposes of the next section, develops Clifford theory for smooth representations of p-adic groups. We recall that Clifford theory in the context of finite groups describes irreducible representations of a finite group G in the presence of a normal subgroup N of G, and takes an especially simple form when N is an abelian normal subgroup, and G can be written as a semi-direct product $N \rtimes H$, see for example, Proposition 25 in [Se]. We have not seen a general form of Clifford theory for smooth representations of a p-adic group, but Bernstein-Zelevinsky in their analysis of representations of $\mathrm{GL}_n(F)$ restricted to a mirabolic subgroup had to develop such a theory — at least in this context — based on rather novel ideas.

Since Bernstein-Zelevinsky's work is written in the specific context of mirabolic subgroups of $\mathrm{GL}_n(F)$, we cannot refer to their theorem, but their method can be adapted to a slightly larger context, which is what we do in this section.

Proposition 118. *Let $G = N \rtimes H$ be a p-adic group with N a finite dimensional vector space over $\mathbb{Q}_p$. Let (π, V) be a smooth representation of G. The group H operates on N, and hence on $\widehat{N}$, the character group of N. Let*

$$\pi_{N,\psi} = \frac{\pi}{\{n \cdot v - \psi(n)v | n \in N, v \in V\}},$$

be the twisted Jacquet module of (π, V). Observe that $\pi_{N,\psi}$ is a module for $N \rtimes H_\psi$ where H_ψ is the stabilizer of ψ in H, and that if $\pi_{N,\psi} \neq 0$, then so is π_{N,ψ^h}, the conjugate of ψ by any $h \in H$. Assume that for $X = \{\psi \in \widehat{N} | \pi_{N,\psi} \neq 0\}$, there are only finitely many orbits of H on X. Then there exists a G-invariant filtration on π whose successive quotients are π_i where the index set $\{i\}$ corresponds to the orbits of H on X, and where the representations π_i are

$$\pi_i = \mathrm{ind}_{N \rtimes H_{\psi_i}}^{N \rtimes H}(\pi_{N,\psi_i}),$$

where H_{ψ_i} is the stabilizer of ψ_i in H. Further, the open orbits of H on X give rise to submodules of π, whereas the closed orbits of H on X give rise to quotient representations of π.

Proof. The main observation of Bernstein-Zelevinsky is that smooth representations of N, a finite dimensional vector space over $\mathbb{Q}_p$, which are described (as for any p-adic group), by nondegenerate representations of the Hecke algebra $\mathcal{H}(N)$, can in this case be described by nondegenerate representations of the algebra of Schwartz functions $\mathbb{S}(\widehat{N})$ on $\widehat{N}$ (an algebra under pointwise multiplication), because of the isomorphism afforded by the Fourier transform :

$$\mathcal{F} : \mathcal{H}(N) \xrightarrow{\cong} \mathbb{S}(\widehat{N}).$$

Now, a nondegenerate representation π of the algebra $\mathbb{S}(\widehat{N})$ gives rise to a sheaf $\mathcal{E}(\pi)$ on $\widehat{N}$ such that $\mathcal{E}_c(\pi)$, the space of compactly supported sections of this sheaf on $\widehat{N}$, is equal to π, and the stalk of the sheaf $\mathcal{E}(\pi)$ at a point $x \in \widehat{N}$ is (cf. Proposition 1.14 of [BZ])

$$\mathcal{E}_x(\pi) = \pi/\{f \cdot v | f \in \mathbb{S}(\widehat{N}) \text{ with } f(x) = 0, v \in \pi\}.$$

Using the identification of $\mathcal{H}(N)$ with $\mathbb{S}(\widehat{N})$, and writing the point $x \in \widehat{N}$ as ψ, it follows that

$$\mathcal{E}_\psi(\pi) = \pi/\{f \cdot v | f \in \mathcal{H}(N) \text{ with } \mathcal{F}(f)(\psi) = \int_N f(y)\psi(y) = 0, v \in \pi\}.$$

Therefore from an application of what is called the lemma of Jacquet-Langlands about Jacquet modules, cf. lemma 2.33 of [BZ], the fiber of $\mathcal{E}$ at a character ψ of N is nothing but the Jacquet module $\pi_{N,\psi}$.

Thus $X = \{\psi \in \widehat{N} | \pi_{N,\psi} \neq 0\}$ is the support of the sheaf $\mathcal{E}(\pi)$.

The sheaf $\mathcal{E}(\pi)$ on $\widehat{N}$ is canonically associated to π, hence π which is actually a representation of $G = N \rtimes H$ but is being considered as a representation of N alone for the moment, becomes a G-equivariant sheaf on $\widehat{N}$.

Given any sheaf $\mathcal{E}$ on a locally compact totally disconnected topological space X with a closed subspace Z, we have the well-known Bernstein-Zelevinsky exact sequence :

$$0 \to \Gamma_c(X - Z, \mathcal{E}) \to \Gamma_c(X, \mathcal{E}) \to \Gamma_c(Z, \mathcal{E}|_Z) \to 0,$$

(where Γ_c refers to compactly supported sections), and this is the exact sequence which is responsible for the filtration on π in the proposition. However, another remark from Bernstein-Zelevinsky is needed before completion of the proof of the proposition, which is that if Z is an orbit of characters of N under H, then $\Gamma_c(Z, \mathcal{E}|_Z)$ can be identified to the induced representation which appears in the statement of the proposition. This is nothing but Proposition 2.23 of [BZ]. $\square$

The following proposition is the exact analogue of Proposition 25 in [Se], a form of Clifford theory, except that our normal abelian subgroup N is more specific than his. (It is actually the previous proposition that we will use in our work.)

Proposition 119. *Let $G = N \rtimes H$ be a p-adic group with N a finite dimensional vector space over $\mathbb{Q}_p$. Let (π, V) be an irreducible smooth representation of G. Then the set of characters $\psi : N \to \mathbb{C}^\times$ of N for which $\pi_{N,\psi} \neq 0$ form a single orbit under H, and*

$$\pi = \mathrm{ind}_{N \rtimes H_\psi}^{N \rtimes H}(\pi_{N,\psi}),$$

where ψ is any character of N for which $\pi_{N,\psi} \neq 0$, and H_ψ is the stabilizer of ψ in H. Further, the representation $\pi_{N,\psi}$ of H_ψ is an irreducible representation, and every irreducible smooth representation of G is obtained in this way.

Proof. The proof of this proposition follows from the observation of Bernstein-Zelevinsky that smooth representations of N, a finite dimensional vector space over $\mathbb{Q}_p$, which are described (as for any p-adic group), by nondegenerate representations of the Hecke algebra $\mathcal{H}(N)$, can in this case be 'geometrized', i.e., correspond to compactly supported global sections of a G-equivariant sheaf of $\mathbb{S}(X)$-modules on a locally compact totally disconnected topological space X. Clearly (compactly supported) global sections of a G-equivariant sheaf $\mathcal{E}$ on a locally compact totally disconnected topological space X gives rise to an irreducible representation of G if and only if,

1. the group G operates transitively on X,
2. the fiber $\mathcal{E}_x$ of the sheaf $\mathcal{E}$ at any point $x \in X$ is an irreducible representation of the stabilizer G_x of the point $x \in X$.

The conclusion of the proposition is now clear. □

4. Non distinction of cuspidal representations

The aim of this section is to prove that cuspidal representations of $\mathrm{U}(n,n)(F)$ are not distinguished by $\mathrm{Sp}(2n,F)$ where F is any non-archimedean local field. The proof of this result — which will assume less than distinction by $\mathrm{Sp}(2n,F)$, and will give more information — will be by an inductive argument on n for which we fix some notation.

Let W_i be the symplectic vector space of dimension $2i$ over F with a fixed basis $\langle e_i, \cdots, e_1, f_1, \cdots, f_i\rangle$ with symplectic form $\langle -, -\rangle$ with the property that $\langle e_j, f_k\rangle = \delta_{jk} = -\langle f_k, e_j\rangle$, and with all the other products zero. The symplectic spaces W_i form a nested sequence of vector spaces with $W_1 \subset W_2 \subset \cdots \subset W_n$.

Given a symplectic space W over F, we have a skew-hermitian space $W_E = W \otimes E$ over E which can be used to define a unitary group $\mathrm{U}(W_E)$ with $\mathrm{Sp}(W) \subset \mathrm{U}(W_E)$.

For $G = \mathrm{Sp}(W)$ (or $\mathrm{U}(W_E)$), define the Klingen parabolic subgroup Q (resp. P) to be the stabilizer of an isotropic line $\langle w\rangle$ in W (resp. W_E). Since any two isotropic vectors in W (or W_E) are conjugate under $\mathrm{Sp}(W)$ (or $\mathrm{U}(W_E)$), the Klingen parabolic subgroups are unique up to conjugacy.

In our analysis below, it will be important to use the subgroup Q^1 of Q (resp. P^1 of P) stabilizing the isotropic vector w itself. We call these subgroups Klingen mirabolic subgroup in analogy with the mirabolic subgroup of Bernstein-Zelevinsky for the group $\mathrm{GL}_n(F)$. They indeed have much in common with the mirabolic subgroup of Bernstein-Zelevinsky. If we denote the Klingen mirabolic in $\mathrm{Sp}(W_n)$ stabilizing the vector $e_n \in W_n$ by Q_n^1, then $Q_n^1 = \mathrm{Sp}(W_{n-1}) \cdot H^{2n-2}(F)$, where $H^{2n-2}(F)$ is the Heisenberg group on the symplectic vector space W_{n-1} (thus $\dim H^{2n-2}(F) = 2n-1$) with the character group of $H^{2n-2}(F)$ identified to W_{n-1} such that the action of $\mathrm{Sp}(W_{n-1})$ on $H^{2n-2}(F)$, and hence on its character group, is the natural action of $\mathrm{Sp}(W_{n-1})$ on W_{n-1}. Similarly, if we denote the Klingen mirabolic in $\mathrm{U}(W_n \otimes E)$ stabilizing the vector $e_n \in W_n \otimes E$ by P_n^1, then $P_n^1 = \mathrm{U}(W_{n-1} \otimes E) \cdot H^{2n-2}(E)$, where $H^{2n-2}(E)$ is the Heisenberg group on the skew-hermitian vector space $W_{n-1} \otimes E$ (thus $\dim H^{2n-2}(E) = 4n-3$) with the character group of $H^{2n-2}(E)$ identified to $W_{n-1} \otimes E$ such that the action of $\mathrm{U}(W_{n-1} \otimes E)$ on $H^{2n-2}(E)$, and hence on its character group, is the natural action of $\mathrm{U}(W_{n-1} \otimes E)$ on $W_{n-1} \otimes E$. An essential input for our proof is the fact that the Heisenberg group $H^{2n-2}(E)$ contains the Heisenberg group $H^{2n-2}(F)$ as a normal subgroup, and their centers are the same, so $H^{2n-2}(E)/H^{2n-2}(F)$ is a vector space over F which is isomorphic to W_{n-1}.

It will be convenient to write out the unipotent radical $N_n(G) = H^{2n-2}(E)$ of P_n^1, as well as the unipotent radical $N_n(S) = H^{2n-2}(F)$ of Q_n^1 both arising as the stabilizer group of the isotropic vector e_n in the matrix form with respect to the ordered basis $\langle e_n, \cdots, e_1, f_1, \cdots, f_n \rangle$ as :

$$N_n(G) = \left\{ \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix}, \begin{array}{ccll} x_i & , & y_i & \in E,\ z \in F \\ x_i & = & \bar{y}_i, & 2 \leq i \leq n-1, \\ x_i & = & -\bar{y}_i, & n \leq i \leq 2n-1. \end{array} \right\}$$

and

$$N_n(S) = \left\{ \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix}, \begin{array}{ccll} & x_i, y_i, z \in F & & \\ x_i & = & y_i, & 2 \leq i \leq n-1, \\ x_i & = & -y_i, & n \leq i \leq 2n-1. \end{array} \right\}$$

Recall that ψ is a fixed non-trivial character of F; assuming that $E = F(\sqrt{d}), d \in F^\times$, let ψ_d be the character on trace zero elements of E defined by $\psi_d(e) = \psi(\sqrt{d}e)$, and let ψ_n be the character of $N_n(G)$:

$$\psi_n \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} = \psi_d(x_{2n-1}+y_{2n-1}) = \psi(\sqrt{d}[x_{2n-1}+y_{2n-1}]),$$

where we note that since $x_{2n-1} = -y_{2n-1}$ for elements in $N_n(S)$, the character ψ_n is trivial on $N_n(S)$, but since $x_{2n-1} = -\bar{y}_{2n-1}$ for elements in $N_n(G)$, $x_{2n-1}+y_{2n-1} = -\bar{y}_{2n-1}+y_{2n-1}$, therefore $\sqrt{d}[x_{2n-1}+y_{2n-1}] \in F$, so the character ψ_n is non-trivial on $N_n(G)$ but trivial on $N_n(S)$.

Proposition 120. *Let π be a smooth representation of the Klingen mirabolic subgroup P_n^1 of $\mathrm{U}(W_n \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$. Then for the unipotent radical $N_n(G)$ of P_n^1, there is a character $\mu : N_n(G) \to \mathbb{C}^\times$ which is either ψ_n or trivial such that π_μ, the maximal quotient of π on which $N_n(G)$ acts by μ is a smooth representation of the Klingen mirabolic subgroup P_{n-1}^1 of $\mathrm{U}(W_{n-1} \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_{n-1}^1 of the symplectic subgroup $\mathrm{Sp}(W_{n-1})$.*

Proof. The proof is a direct consequence of the Clifford theory developed in the last section. Recall that we have denoted by $N_n(S)$ (resp. $N_n(G)$), the unipotent radical of the Klingen mirabolic in $\mathrm{Sp}(W_n)$ (resp. $\mathrm{U}(W_n \otimes E)$) stabilizing the isotropic vector e_n. Both $N_n(S)$ and $N_n(G)$ are normalized by $\mathrm{Sp}(W_{n-1})$, and $N_n(S)$ is contained in $N_n(G)$ as a normal subgroup with $N_n(G)/N_n(S) \cong W_{n-1}$ as a module for $\mathrm{Sp}(W_{n-1})$.

Let $\pi_{N_n(S)}$ be the largest quotient of π on which $N_n(S)$ operates trivially. It is a smooth module for $\mathrm{Sp}(W_{n-1}) \ltimes N_n(G)/N_n(S) \cong \mathrm{Sp}(W_{n-1}) \ltimes W_{n-1}$. Since π is distinguished by the mirabolic subgroup Q_n^1 of $\mathrm{Sp}(W_n)$, $\pi_{N_n(S)}$ is distinguished by $\mathrm{Sp}(W_{n-1})$. Since $\pi_{N_n(S)}$ is a module for $\mathrm{Sp}(W_{n-1}) \ltimes W_{n-1}$, we can apply Clifford theory to understand this as a module for $\mathrm{Sp}(W_{n-1})$.

The action of $\mathrm{Sp}(W_{n-1})$ on the character group of W_{n-1} has two orbits, one consisting of the trivial character, and the other passing through the character ψ_n whose stabilizer in $\mathrm{Sp}(W_{n-1})$ is the Klingen mirabolic subgroup Q_{n-1}^1 of the symplectic subgroup $\mathrm{Sp}(W_{n-1})$. Notice that the character ψ_n of $N_n(G)/N_n(S)$ can also be considered as a character of $N_n(G)$, and the stabilizer of this character of $N_n(G)$ in $\mathrm{U}(W_{n-1} \otimes E)$ is the Klingen mirabolic subgroup P_{n-1}^1 of $\mathrm{U}(W_{n-1} \otimes E)$.

By Clifford theory, $\pi_{N_n(S)}$ as a module for $\mathrm{Sp}(W_{n-1})$ has two sub-quotients corresponding to the two orbits for the action of $\mathrm{Sp}(W_{n-1})$ on the character group of $N_n(G)/N_n(S) \cong W_{n-1}$. The sub-quotient corresponding to the trivial representation of W_{n-1} being $\pi_{N_n(G)}$, and the other subquotient (in fact a sub-module) being

$$\mathrm{ind}_{Q_{n-1}^1}^{\mathrm{Sp}(W_{n-1})}(\pi_{\psi_n})$$

where π_{ψ_n} is the maximal quotient of π on which $N_n(G)$ acts by ψ_n.

Since $\pi_{N_n(S)}$ is distinguished by $\mathrm{Sp}(W_{n-1})$, one of these two sub-quotients is distinguished by $\mathrm{Sp}(W_{n-1})$, hence by Frobenius reciprocity either $\pi_{N_n(G)}$ is distinguished by $\mathrm{Sp}(W_{n-1})$ and therefore also by its Klingen mirabolic subgroup, or π_{ψ_n} which is a smooth representation of the Klingen mirabolic subgroup P_{n-1}^1 of $\mathrm{U}(W_{n-1} \otimes E)$ is distinguished by the Klingen mirabolic subgroup Q_{n-1}^1 of the symplectic subgroup $\mathrm{Sp}(W_{n-1})$, completing the proof of the proposition. $\square$

Corollary 121. *A smooth representation π of the Klingen mirabolic subgroup P_n^1 of $\mathrm{U}(W_n \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ carries a nonzero μ_n-linear form for the group of the upper-triangular unipotent matrices in $\mathrm{U}(W_n \otimes E)$ for μ_n given by :*

$$\mu_n \begin{pmatrix} 1 & x_1 & * & * & * & * \\ 0 & 1 & x_2 & * & * & * \\ 0 & 0 & 1 & x_3 & * & * \\ 0 & & & \ddots & \ddots & \vdots \\ 0 & & & & 1 & x_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} = \psi_d(\epsilon_1[x_1+x_{2n-1}]+\epsilon_2[x_2+x_{2n-2}]+\cdots+\epsilon_{n-1}[x_{n-1}+x_{n+1}]),$$

where the ϵ_i are either 0 or 1, and we note the most important aspect of this character that the term x_n is missing on the right. (Recall that $\psi_d(e) = \psi(\sqrt{d}e)$ is a character on trace zero elements of E.)

Proof. Assuming the corollary for $n-1$, it is an immediate consequence of the proposition that it holds for n with $\epsilon_1 = 0$ or 1 depending on the two cases in the proposition ; note also that for $n = 1$, the Klingen mirabolic subgroup of both $\mathrm{Sp}(W_1)$ and $\mathrm{U}(W_1 \otimes E)$ is the group of 2×2 upper triangular matrices with entries in F for which the corollary is obvious. The form of the character μ_n follows from the form of the character ψ_n defined before Proposition 120. □

Corollary 122. *Any representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ is a sub-quotient of a principal series representation of $\mathrm{U}(n,n)(F)$ induced from the Siegel parabolic (with Levi $\mathrm{GL}_n(E)$). In particular, a representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ cannot be cuspidal.*

Proof. A representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ is *a fortiori* distinguished by the Klingen mirabolic in $\mathrm{Sp}_{2n}(F)$. It suffices then to observe that the character appearing in Corollary 121 above is trivial on the unipotent radical of the Siegel parabolic, hence the Jacquet module corresponding to the Siegel parabolic is nonzero. □

Corollary 123. *For any representation π of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$, there is a character $\psi : U \to \mathbb{C}^\times$ of the unipotent radical U of a minimal parabolic for which $\pi_{U,\psi} \neq 0$.*

(By a theorem of Zelevinsky, any representation of $\mathrm{GL}_{2n}(F)$ has this property, but this is not the case for other groups, not even for Unitary groups.)

Remark 16. In this section we have not used any property of p-adic fields, and thus the results in this section remain valid for finite fields. In Theorem 2.2.1 of [He], Henderson has given a complete classification of representations of $\mathrm{U}_{2n}(\mathbb{F}_q)$ which are distinguished by $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, in particular he proves that there are no cuspidal representations of $\mathrm{U}_{2n}(\mathbb{F}_q)$ which are distinguished by $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.

Remark 17. The proof given here on distinction of representations of $\mathrm{U}(n,n)(F)$ by $\mathrm{Sp}_{2n}(F)$ remains valid almost verbatim for representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ giving another proof of the theorem of Heumos-Rallis in [HR] on non-existence of cuspidal representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. In fact the proof given here uses just the Klingen mirabolic subgroup of $\mathrm{Sp}_{2n}(F)$ to draw this conclusion, and therefore cannot be expected to give the much finer results which have become available on representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. However, note that our proofs use more of $\mathrm{Sp}_{2n}(F)$, and its Klingen mirabolic subgroup, and almost nothing about the ambient group $\mathrm{U}(n,n)(F)$, or in this case, $\mathrm{GL}_{2n}(F)$, and therefore, in particular our proofs work as well to understand representations of $\mathrm{SL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. We only state the following proposition in this regard.

Proposition 124. *A smooth representation π of $\mathrm{SL}_{2n}(F) = \mathrm{SL}(W_n)$ which is distinguished by the symplectic subgroup $\mathrm{Sp}(W_n)$ carries a nonzero μ_n-linear form for the group of the upper-triangular unipotent matrices in $\mathrm{SL}(W_n)$ for μ_n given by :*

$$\mu_n \begin{pmatrix} 1 & x_1 & * & * & * & * \\ 0 & 1 & x_2 & * & * & * \\ 0 & 0 & 1 & x_3 & * & * \\ 0 & & & \ddots & \ddots & \vdots \\ 0 & & & & 1 & x_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} = \psi(\epsilon_1[x_1+x_{2n-1}]+\epsilon_2[x_2+x_{2n-2}]+\cdots+\epsilon_{n-1}[x_{n-1}+x_{n+1}]),$$

where the ϵ_i are either 0 or 1, and ψ is any (fixed) nontrivial character of F.

We next recall from Zelevinsky [Ze] the notion of degenerate Whittaker model of an arbitrary irreducible smooth representation π of $\mathrm{GL}_n(F)$. He defines in §8 of [Ze] a character θ on the group U of upper triangular unipotent elements of $\mathrm{GL}_n(F)$ by

$$\theta(u_{ij}) = \psi(\textstyle\sum u_{i,i+1}),$$

where $\sum$ runs over all integers $1, 2, \cdots, n-1$ except,

$$n-\lambda_1, n-\lambda_1-\lambda_2, \cdots, n-\lambda_1-\lambda_2-\cdots-\lambda_{k-1},$$

where the integers λ_i are inductively defined with λ_1 being the highest nonzero derivative of π, λ_2 the highest nonzero derivative of π^{λ_1}, and so on. It is a theorem of Zelevinsky (corollary in §8.3 of [Ze]) that there is a linear form $\ell : \pi \to \mathbb{C}$ on which the group U of upper triangular unipotent matrices acts by the character θ, and the space of such linear forms has dimension 1.

Conjecture 1. *Let π be an irreducible admissible representation of $\mathrm{GL}(W_n)$ which is distinguished by $\mathrm{Sp}(W_n)$. Write π restricted to $\mathrm{SL}(W_n)$ as a sum of irreducible* *representations $\pi = \sum \pi_\alpha$ (with multiplicity 1). Then exactly one of the representations π_α is distinguished by $\mathrm{Sp}(W_n)$, and the one which is distinguished by $\mathrm{Sp}(W_n)$ is the one which carries the invariant linear form θ of Zelevinsky defined above. (There is a unique representation of $\mathrm{SL}(W_n)$ carrying the invariant linear form θ by the multiplicity one assertion of Zelevinsky for the group $\mathrm{GL}_n(F)$.)*

Remark 18. From the classification due to Offen-Sayag of irreducible admissible representations of $\mathrm{GL}(W_n)$ which are distinguished by $\mathrm{Sp}(W_n)$, which we will recall in section 9, it follows that the character θ of Zelevinsky is of the form μ_n introduced in Corollary 121. Further, observe that the choice of the character ψ in Conjecture 1 is not relevant since conjugation by the diagonal matrix

$$a_t = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^n \end{pmatrix}$$

is scaling by t on all simple root spaces except the 'middle' one (which is not there in μ_n), so acts transitively on the set of characters μ_n arising out of different choices of ψ, and a_t being in $\mathrm{GSp}(W_n)$, it preserves distinction by $\mathrm{Sp}(W_n)$.

Proposition 125. *Conjecture 1 is true for the Speh module $Sp_m(\pi)$ (where π is a cuspidal representation of $\mathrm{GL}_d(F)$ and m is even, so that $Sp_m(\pi)$ has symplectic model) which is the unique irreducible quotient of the principal series representation $\pi \cdot \nu^{(m-1)/2} \times \cdots \times \pi \cdot \nu^{-(m-1)/2}$ of $\mathrm{GL}_{md}(F)$.*

Proof. It is known that for the Speh module $Sp_m(\pi)$, the integers λ_i introduced above are all equal to d, and $k = m$. Thus the character θ of Zelevinsky is the character of the group U of upper triangular unipotent matrices given by

$$\theta(u_{ij}) = \psi(\sum u_{i,i+1}),$$

where $\sum$ runs over all integers $1, 2, \cdots, n-1$ except,

$$n-d, n-2d, \cdots, n-(m-1)d = d.$$

The main point about the Speh module $Sp_m(\pi)$, which we will presently prove, being that θ is the only character (up to conjugacy) of the unipotent group U for which there is a θ-invariant linear form. Thus the only character which appears in Proposition 124 is θ, proving conjecture 1 for the Speh modules $Sp_m(\pi)$.

To prove the assertion regarding characters of U appearing in the Speh module $Sp_m(\pi)$, note that any character of U is of the form

$$\theta_S(u_{ij}) = \psi(\sum a_i u_{i,i+1}),$$

where $a_i \in F$, and S is defined to be the set of integers i for which $a_i = 0$. Construct the standard parabolic $P = M_S N_S$ of $\mathrm{GL}_n(F)$ such that the only simple root spaces in N are α_i for $i \in S$. The character θ_S is clearly trivial on N_S, and therefore the Jacquet module of π with respect to N_S is nonzero, and is in fact generic. Now we appeal to the 'hereditary' property of Jacquet modules for Speh modules : that the Jacquet modules of $Sp_m(\pi)$ are themselves product of Speh modules on π, and therefore the only nonzero generic Jacquet module corresponds to the partition $(d, d, \cdots, d)$ of md, proving the assertion on the characters of U appearing in the Speh module $Sp_m(\pi)$. $\square$

Remark 19. In this final remark of the section, we try to delineate the 'group theory' which goes into the proof of the main result, Proposition 120. The paper [AGR] calls a pair (G, H) a vanishing pair, if there are no cuspidal representations of G distinguished by H. In this paper we have proved that $(\mathrm{U}_{2n}, \mathrm{Sp}_{2n})$ is a vanishing pair. How did we achieve it ? To simplify language, let's be in the context of algebraic groups over finite fields. We need to use the subgroup H to construct the unipotent radical N of a parabolic in G such that a cuspidal representation π of G distinguished by H is also distinguished by N leading to a contradiction to cuspidality of π. Well, begin with the unipotent radical $N(H)$ of a parabolic in H. Take its normalizer $P_G(N)$ in G, and let $N(G)$ be the unipotent radical of $P_G(N)$, which clearly contains $N(H)$ as a normal subgroup. Since the representation π we are considering has a H fixed vector, it certainly has $N(H)$ fixed vectors, and $\pi^{N(H)}$ is a module for $P_G(N)/N(H)$. In our case, $N(G)/N(H)$ was an abelian group, allowing us to understand $\pi^{N(H)}$ as a module for $P_G(N)/N(H)$, in particular also for $N(G)$. The group $N(G)$ is nearer to the unipotent radical of a parabolic in G (this is a general theorem of Borel-Tits of going from any unipotent group in G to the unipotent radical of a parabolic in G by an iterative process of the above kind). We do not quite get distinction by $N(G)$, but by a character χ of $N(G)/N(H)$, whose kernel $\ker(\chi)$ is a codimension one subspace of $N(G)$ (containing $N(H)$), so we are making progress. The rep'n $\pi^{N(H)}$ of $P_G(N)/N(H)$ is distinguished by $P_G(N) \cap H$. This allows one to get some more unipotents from H to be augmented to $\ker(\chi)$ to reach towards the desired unipotent radical N of a parabolic in G.

5. Non distinction of Cuspidal automorphic representations

In this section we prove that for cuspidal automorphic functions f on $\mathrm{U}(n,n)(\mathbb{A}_k)$ we must have :

$$\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(h)dh = 0.$$

Actually we first prove what appears to be a stronger result, that the period integral of cuspidal automorphic functions f on $\mathrm{U}(n,n)(\mathbb{A}_k)$ on Klingen mirabolic Q_n^1 of Sp_{2n} is zero :

$$\int_{Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)} f(h)dh = 0;$$

however, vanishing of the symplectic period is not a formal consequence of this. For our local theorem, this was no issue : if there are no invariant linear forms for the Klingen mirabolic, a fortiori, there are none for the larger symplectic group. In the global situation, because we are dealing with integration on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$ versus integration on $Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)$, we are not quite in a context to be able to use Fubini's theorem, and such a conclusion is not obvious, and is effected using an Eisenstein series, a trick that we learnt from [AGR]. (This trick is the global analogue of the identity in the context of finite or p-adic groups : $\pi \otimes \mathrm{ind}_H^G 1 = \mathrm{ind}_H^G(\pi|_H)$ for π a representation of G, which is a complicated way of saying that if π carries a G-invariant linear form, then $\pi|_H$ carries an H-invariant linear form.)

The proof of vanishing of period integral on Klingen mirabolic, will follow closely our local proof. We will also follow exactly the same notation as there, thus W_i will be symplectic vector space over k with basis $\langle e_i, \cdots, e_1, f_1, \cdots, f_i\rangle$ with the symplectic form $\langle -,-\rangle$ with the property that $\langle e_j, f_k\rangle = \delta_{jk} = -\langle f_k, e_j\rangle$, and with all the other products zero. The symplectic spaces W_i form a nested sequence of vector spaces with $W_1 \subset W_2 \subset \cdots \subset W_n$. Given a symplectic space W over k, and K/k a quadratic extension, we have a skew-hermitian space $W_K = W \otimes K$ over K which can be used to define a unitary group $\mathrm{U}(W_K)$ with $\mathrm{Sp}(W) \subset \mathrm{U}(W_K)$.

We begin with an analogue of Clifford theory. In fact, in the local theory, one could separate the role of Clifford theory, Mackey theory and the Frobenius reciprocity, which together allow one to understand when a representation of $G = A \rtimes H$ has an H-invariant linear form. In the global context, the three steps will merge into one, and we will directly find when an automorphic representation of G has nonzero period integral along H.

Let $G = A \rtimes H$ be a semi-direct product of algebraic groups over a number field k where $A \cong k^d$ for some integer d. Fix $\psi_0 : \mathbb{A}_k/k \to \mathbb{C}^\times$ to be a nontrivial character. For any linear map $\ell : A \to k$, we get an automorphic character $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$, and all automorphic characters on $A(\mathbb{A}_k)/A(k)$ are of this form; thus automorphic characters on $A(\mathbb{A}_k)/A(k)$ are in bijective correspondence with the dual vector space $A^\vee(k)$ of the vector space A over k.

Let H_ℓ be the stabilizer in H of a linear map $\ell : A \to k$. Then H_ℓ is an algebraic subgroup of H defined over k such that $H_\ell(k) = H_\psi(k)$ is the stabilizer of the automorphic character $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$. We will assume in what follows that $H(k)\backslash H(\mathbb{A}_k)$, as well as $H_\psi(k)\backslash H_\psi(\mathbb{A}_k)$ have finite measures for all characters $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$.

For a function f on $G(k)\backslash G(\mathbb{A}_k)$, define its Fourier coefficient f_ψ to be the function on $H_\psi(k)\backslash H_\psi(\mathbb{A}_k)$ defined by :

$$f_\psi(h) = \int_{A(\mathbb{A}_k)/A(k)} f(ah)\psi(a)da,$$

where da is a Haar measure on $A(\mathbb{A}_k)/A(k)$. Taking Fourier coefficients gives an $H_\psi(\mathbb{A}_k)$-equivariant map from smooth functions on $G(k)\backslash G(\mathbb{A}_k)$ to smooth functions on $H_\psi(k)\backslash H_\psi(\mathbb{A}_k)$:

$$\mathcal{F}_\psi : C^\infty(G(k)\backslash G(\mathbb{A}_k)) \to C^\infty(H_\psi(k)\backslash H_\psi(\mathbb{A}_k)).$$

It will be important to note that $\mathcal{F}_\psi$ takes bounded functions in $C^\infty(G(k)\backslash G(\mathbb{A}_k))$ to bounded functions in $C^\infty(H_\psi(k)\backslash H_\psi(\mathbb{A}_k))$. Since $\mathcal{F}_\psi$ commutes with $H_\psi(\mathbb{A}_k)$, if we have a space π of bounded functions $C^\infty(G(k)\backslash G(\mathbb{A}_k))$ invariant under differential operators coming from $G(k \otimes \mathbb{R})$, in particular from $H_\psi(k \otimes \mathbb{R})$, the image of $\mathcal{F}_\psi$ under π will consist of bounded functions in $C^\infty(H_\psi(k)\backslash H_\psi(\mathbb{A}_k))$ invariant under differential operators coming from $H_\psi(k \otimes \mathbb{R})$.

Proposition 126. *With the notation as above (in particular $G = A \rtimes H$, a semi-direct product of algebraic groups over a number field k with A a vector space over k, and $H_\psi(k)\backslash H_\psi(\mathbb{A}_k)$ have finite measures for all characters $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$), let π be a space of smooth functions on $G(k)\backslash G(\mathbb{A}_k)$ which is $G(\mathbb{A}_k)$-invariant, and consists of bounded functions such that the period integral on $H(k)\backslash H(\mathbb{A}_k)$ is not identically zero. Then there is a function $f \in \pi$, and a character $\ell : A \to k$ for which the Fourier coefficient $f_\ell = f_\psi$ defined above to be a function on $H_\psi(k)\backslash H_\psi(\mathbb{A}_k)$ has nonzero period integral on $H_\psi(k)\backslash H_\psi(\mathbb{A}_k)$.*

Proof. Let's begin with the Fourier expansion :

$$f(ah) = \sum_{\psi : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times} f_\psi(h)\psi(a),$$

where ψ runs over all automorphic characters $\psi : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$ which as noted earlier all arise as $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$ for a linear map $\ell : A \to k$.

Evaluating the Fourier expansion at $a = 1$,

$$f(h) = \sum_\psi f_\psi(h),$$

hence,

$$\begin{aligned}\int_{H(k)\backslash H(\mathbb{A}_k)} f(h)dh &= \int_{H(k)\backslash H(\mathbb{A}_k)} \sum_\psi f_\psi(h)dh \\ &= \sum_\psi \int_{H(k)\backslash H(\mathbb{A}_k)} f_\psi(h)dh.\end{aligned}$$

We need to justify interchanging summation and integration above which we shall do separately in the next two Lemmas so as not to disrupt the flow of argument here.

Combining characters $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \to \mathbb{C}^\times$ for a linear map $\ell : A \to k$ which are in one orbit for $H(k)$ — the set of $H(k)$ orbits of such characters being the quotient set $A^\vee(k)/H(k)$ — we find that :

$$\begin{aligned}\int_{H(k)\backslash H(\mathbb{A}_k)} f(h)dh &= \sum_\psi \int_{H(k)\backslash H(\mathbb{A}_k)} f_\psi(h)dh \\ &= \sum_{\psi \in A^\vee(k)/H(k)} \int_{H_\psi(k)\backslash H(\mathbb{A}_k)} f_\psi(h)dh \\ &= \sum_{\psi \in A^\vee(k)/H(k)} \int_{H_\psi(\mathbb{A}_k)\backslash H(\mathbb{A}_k)} \left[\int_{H_\psi(k)\backslash H_\psi(\mathbb{A}_k)} f_\psi(hh')dh\right] dh.'\end{aligned}$$

Therefore if the period integral on $H(k)\backslash H(\mathbb{A}_k)$ is nonzero, so must the inner integral

$$\int_{H_\psi(k)\backslash H_\psi(\mathbb{A}_k)} f_\psi(hh')dh,$$

too for some $h' \in H(\mathbb{A}_k)$ and some automorphic character ψ on $A(k)\backslash A(\mathbb{A}_k)$. Since the space of functions in π is right invariant under $H(\mathbb{A}_k)$, this proves the proposition. $\square$

The following two lemmas justify interchanging summation and integration used above.

Lemma 127. *Suppose $f(x,t)$ is a function on $X \times T = X \times (\mathbb{R}/\mathbb{Z})^d$ where X is a measure space. Assume that $f(x,t)$ is infinitely differentiable as a function of $t \in T$, for all $x \in X$, $t \in (\mathbb{R}/\mathbb{Z})^d$. Assume that f as well as all its derivatives (with constant coefficients) along $(\mathbb{R}/\mathbb{Z})^d$ are bounded as a function on $X \times T$, and that X has finite measure. Then $\sum_{\underline{n}} \int_X f_{\underline{n}}(x)dx$ is an absolutely convergent series, and*

$$\int_X f(x)dx = \sum_{\underline{n}} \int_X f_{\underline{n}}(x)dx,$$

where for $\underline{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, $\underline{t} = (t_1, \cdots, t_d) \in (\mathbb{R}/\mathbb{Z})^d = T$, $f_{\underline{n}}(x)$ is the $\underline{n}$-th

Fourier coefficient defined by :

$$f_{\underline{n}}(x) = \int_{(\mathbb{R}/\mathbb{Z})^d} f(x,\underline{t}) e^{2\pi i(\sum_k n_k t_k)} dt_1 \cdots dt_d.$$

Proof. We give a proof only for $d = 1$. By the Cauchy-Schwarz inequality, we have :

$$\sum_{n \neq 0} |a_n| \leq \sum_{n \neq 0} |na_n|^2 \sum_{n \neq 0} \frac{1}{n^2}.$$

If a_n are the Fourier coefficients of a function $f(t)$ on $\mathbb{R}/\mathbb{Z}$, na_n are the Fourier coefficients of the function $\frac{df}{dt}$. Therefore using Parseval's (= Plancherel) theorem,

$$\begin{aligned}
\int_X (\sum_{n \neq 0} |f_n(x)|) dx &\leq \frac{\pi^2}{3} \int_X (\sum_{n \neq 0} |nf_n(x)|^2) dx \\
&\leq \int_X \left| \frac{df}{dt}(x,t) \right|^2 dx dt \\
&< \infty,
\end{aligned}$$

where the last conclusion is arrived at because $\frac{df}{dt}$ is a bounded function on $X \times T$, and X has finite measure.

Finally, because X is assumed to have finite measure, the Lebesgue dominated convergence theorem allows us to interchange summation and integration above. □

Following is the adelic analogue of the previous lemma which can be easily deduced from it, but we shall not do so here. In this lemma, we will use the standard notion of a 'smooth' function on $\mathbb{A}_k^d$ built out of characteristic functions of (translates of) compact open subgroups of finite part of the adele group, and smooth functions at infinity.

Lemma 128. *Suppose $f(x,t)$ is a function on $X \times T = X \times (k \backslash \mathbb{A}_k)^d$ where X is a measure space. Assume that $f(x,t)$ is 'smooth' as a function of $t \in \mathbb{A}_k^d$, for all $x \in X$, $t \in \mathbb{A}_k^d$. Assume that f as well as all its derivatives (with constant coefficients) along $(k \backslash \mathbb{A}_k)^d$ are bounded as a function on $X \times (k \backslash \mathbb{A}_k)^d$, and that X has finite measure. Then $\sum_{y \in k^d} \int_X f_y(x) dx$ is an absolutely convergent series, and*

$$\int_X f(x) dx = \sum_{y \in k^d} \int_X f_y(x) dx,$$

where for $y = (y_1, \cdots, y_d) \in k^d$, $\underline{t} = (t_1, \cdots, t_d) \in \mathbb{A}_k^d$, $f_y(x)$ is the Fourier coefficient defined by :

$$f_y(x) = \int_{(k \backslash \mathbb{A}_k)^d} f(x,\underline{t}) \psi(\sum_k y_k t_k) dt_1 \cdots dt_d.$$

The purpose of the following lemma and its corollary is to have the most obvious relationship between period integrals on a group and any of its normal subgroups;

Lemma 129. *Let G be an algebraic group over a number field k, and N a normal algebraic subgroup of G defined over k. Then for L^1-functions f on $G(k)\backslash G(\mathbb{A}_k)$, and for an appropriate choice of right Haar measures, we have,*

$$\int_{G(k)\backslash G(\mathbb{A}_k)} f(g)dg = \int_{(N\backslash G)(k)\backslash (N\backslash G)(\mathbb{A}_k)} \left[\int_{N(k)\backslash N(\mathbb{A}_k)} f(n\bar{g})dn\right] d\bar{g}.$$

Corollary 3. *Let G be an algebraic group over a number field k, and N a normal algebraic subgroup of G defined over k. Then for a space V of L^1-functions on $G(k)\backslash G(\mathbb{A}_k)$ which is invariant under right translations by $G(\mathbb{A}_k)$, if the period integral on $G(k)\backslash G(\mathbb{A}_k)$ is not identically zero on V, then the period integral on $N(k)\backslash N(\mathbb{A}_k)$ is also not identically zero on V.*

The following proposition is the exact global analogue of Proposition 120 of the last section, with a proof which is almost verbatim the proof there. The notation used in this proposition is accordingly the same as there, in particular we remind the reader of the character ψ_n introduced before Proposition 120.

Proposition 130. *Let Π be a space of bounded smooth functions on $P_n^1(k)\backslash P_n^1(\mathbb{A}_k)$ which is invariant under $P_n^1(\mathbb{A}_k)$ where P_n^1 is the Klingen mirabolic subgroup of $\mathrm{U}(W_n \otimes K)$ consisting of cuspforms (for 'standard' parabolics contained in $\mathrm{U}(W_n \otimes K)$: notice that even if a standard parabolic is not contained in $P_n^1(\mathbb{A}_k)$, its unipotent radical is). Assume that the period integral of Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ is not identically zero. Then for the unipotent radical $N_n(G)$ of P_n^1, and the automorphic character $\psi_n : N_n(G)(k)\backslash N_n(G)(\mathbb{A}_k) \to \mathbb{C}^\times$, the image of Π under the Fourier coefficient map $\mathcal{F}$ introduced above, is a nonzero representation of $P_{n-1}^1(\mathbb{A}_k)$ for P_{n-1}^1 the Klingen mirabolic subgroup of $\mathrm{U}(W_{n-1} \otimes K)$ consisting of bounded cuspforms for which the period integral on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ is not identically zero.*

Proof. Let $N_n(S)$ be the unipotent radical of Q_n^1, and $N_n(G)$ the unipotent radical of P_n^1. Since $N_n(S)$ is a normal subgroup of Q_n^1, and we are given that Π has nonzero period integral on $Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)$, it follows from Corollary 3 that the period integral on $N_n(S)(k)\backslash N_n(S)(\mathbb{A}_k)$ is also nonzero. We consider the trivial Fourier coefficient of Π with respect to $N_n(S)(k)\backslash N_n(S)(\mathbb{A}_k)$, to construct a space of functions — call it $\Pi_{N_n(S)}$ — on $\mathrm{Sp}_{2n-2}(\mathbb{A}_k) \ltimes N_n(G)/N_n(S)(\mathbb{A}_k) = \mathrm{Sp}_{2n-2}(\mathbb{A}_k) \ltimes \mathbb{A}_k^{2n-2}$. We now apply Proposition 126 with $G = H \ltimes A = \mathrm{Sp}_{2n-2} \ltimes k^{2n-2}$ and $\pi = \Pi_{N_n(S)}$, observing that the character ψ_n is trivial on $N_n(S)(\mathbb{A}_k)$, therefore it defines a character of $A(\mathbb{A}_k) = N_n(S)(\mathbb{A}_k)\backslash N_n(G)(\mathbb{A}_k)$, and the corresponding Fourier coefficient on G is the same as that on P_n^1 because of :

$$\int_{N_n(G)(k)\backslash N_n(G)(\mathbb{A}_k)} f(n)\psi_n(n)dn = \int_{A(k)\backslash A(\mathbb{A}_k)} \left[\int_{N_n(S)(k)\backslash N_n(S)(\mathbb{A}_k)} f(nn')dn\right] \psi_n(n')dn'.$$

The image of Π under the Fourier coefficient map $\mathcal{F}$ introduced above consists of cuspforms is an easy result which we leave to the reader; boundedness of functions in $\mathcal{F}(\Pi)$ is clear. $\square$

Proposition 131. *Let* Π *be a cuspidal automorphic representation of* $U(W_n \otimes K)$*, and let* Q_n^1 *be the Klingen mirabolic subgroup in* $\mathrm{Sp}(W_n)$*. Then if* $\int_{Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)} f(g)dg$ *vanishes for all* $f \in \Pi$*,* $\int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)dg$ *vanishes too for all* $f \in \Pi$*.*

Proof. Assuming that $\int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)dg \neq 0$ for some $f \in \Pi$, we shall prove by contradiction that $\int_{Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)} f(g)dg \neq 0$ also for some $f \in \Pi$. Assume if possible that $\int_{Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)} f(g)dg$ vanishes for all $f \in \Pi$.

Let $I(s) = \mathrm{Ind}_{Q_n(\mathbb{A}_k)}^{\mathrm{Sp}_{2n}(\mathbb{A}_k)}(\delta^s)$ be the principal series representation of $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ for δ 'half the sum of positive roots' for the Klingen parabolic $Q_n(\mathbb{A}_k)$. If we write the natural decomposition of Q_n as $Q_n = \mathbb{G}_m \times Q_n^1$, then for $(t,q) \in Q_n(\mathbb{A}_k) = \mathbb{A}_k^\times \times Q_n^1(\mathbb{A}_k)$, $\delta(t,q) = |t|^n$ where $|t|$ is the usual absolute value on $\mathbb{A}_k^\times$.

Let $\phi(g,s) \in I(s)$ be a 'standard' section of this analytic family of principal series representations, thus for each $s \in \mathbb{C}$, $\phi(g,s)$ are functions on $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ such that,

$$\phi(pg,s) = |p|^s\phi(g,s),$$

where $p = (t,q) \in Q_n(\mathbb{A}_k) = \mathbb{A}_k^\times \times Q_n^1(\mathbb{A}_k)$, and $|p|^s = |t|^{ns}$. Let $K_{\mathbb{A}}$ be the maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ given by $K_{\mathbb{A}} = \prod_{v<\infty} \mathrm{Sp}_{2n}(\mathcal{O}_v) \times K_\infty$ so that $\mathrm{Sp}_{2n}(\mathbb{A}_k) = K_{\mathbb{A}} \times Q_n(\mathbb{A})$. To say that a family of functions $\phi(g,s) \in I(s)$ is a 'standard' section means that its restriction to $K_{\mathbb{A}}$ is a smooth function independent of s. By the transformation property, $\phi(pg,s) = |p|^s\phi(g,s)$, the restriction of $\phi(g,s)$ to $K_{\mathbb{A}}$ has the property that

$$\phi(pg,s) = \phi(g,s),$$

for all $p \in Q_n(K_{\mathbb{A}}) = K_{\mathbb{A}} \cap Q_n(\mathbb{A})$. Conversely, given a smooth function ϕ on $K_{\mathbb{A}}$ with the property $\phi(pg) = \phi(g)$ for all $p \in Q_n(K_{\mathbb{A}}) = K_{\mathbb{A}} \cap Q_n(\mathbb{A})$, there is a unique standard section $\phi(g,s) \in I(s)$. In particular, there is the unique section $\phi_0(g,s)$ which is identically 1 on $K_{\mathbb{A}}$, which may be called the standard spherical section of the family $I(s)$.

Now, for a standard section $\phi(g,s) \in I(s)$, consider the Eisenstein series

$$E(\phi,g,s) = \sum_{\gamma \in Q_n(k)\backslash \mathrm{Sp}_{2n}(k)} \phi(\gamma g,s),$$

a meromorphic family of functions on $\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$ which is known to be absolutely convergent for $Re(s) > 1$, which for the function $\phi(g,s) = \phi_0(g,s)$

has a simple pole at $s = 1$, and $\mathrm{Res}_{s=1}E(\phi_0, g, s)$ is the constant function 1 on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$. In what follows, we shall denote by $E(g, s)$, the Eisenstein series $E(\phi_0, g, s)$.

As is well known, a cuspform is rapidly decreasing, and an Eisenstein series is slowly increasing. It follows that the product of a cusp form (on a group G restricted to a subgroup H) with an Eisenstein series on H is still rapidly decreasing, and therefore for f any function in Π restricted to $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$, it is meaningful to integrate $f \cdot E(g, s)$ on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$, and unfold the Eisenstein series :

$$\begin{aligned} \int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)E(g,s)dg &= \int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) \sum_{\gamma\in Q_n(k)\backslash\mathrm{Sp}_{2n}(k)} \phi_0(\gamma g, s)dg \\ &\overset{(*)}{=} \int_{Q_n(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)\phi_0(g,s)dg. \end{aligned}$$

Using the decomposition, $\mathrm{Sp}_{2n}(\mathbb{A}_k) = K_{\mathbb{A}} \times Q_n(\mathbb{A}) = K_{\mathbb{A}} \times \mathbb{A}_k^\times \times Q_n^1(\mathbb{A})$, we write the Haar measure on $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ as $dg = dkdq = dkd^\times adq'$, so that for any L^1 function λ on $\mathrm{Sp}_{2n}(\mathbb{A}_k)$, we have the following form of Fubini's theorem :

$$\begin{aligned} \int_{\mathrm{Sp}_{2n}(\mathbb{A}_k)} \lambda(g)dg &= \int_{Q_n(K_{\mathbb{A}})\backslash K_{\mathbb{A}}} \int_{Q_n(\mathbb{A}_k)} \lambda(k,q)dqdk \\ &= \int_{Q_n(K_{\mathbb{A}})\backslash K_{\mathbb{A}}} \int_{\mathbb{A}_k^\times} \left[\int_{Q_n^1(\mathbb{A}_k)} \lambda(k,a,q')dq'\right] d^\times adk. \end{aligned}$$

From the equation $(*)$ on noting that $\phi_0(g, s) = 1$ on $K_{\mathbb{A}}$, it follows that

$$\begin{aligned} \int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)E(g,s)dg &= \int_{Q_n(K_{\mathbb{A}})\backslash K_{\mathbb{A}}} \left[\int_{\mathbb{A}_k^\times/k^\times} \int_{Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)} |a|^s f(q'ak)dq'd^\times a\right] dk \\ &= \int_{Q_n(K_{\mathbb{A}})\backslash K_{\mathbb{A}}} \left[\int_{\mathbb{A}_k^\times/k^\times} |a|^s F(a,k)d^\times a\right] dk, \end{aligned}$$

where

$$F(a,k) = \int_{Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)} f(q'ak)dq',$$

is the integral of a bounded function on a space with finite measure, so the integral is absolutely convergent. Further, $F(a, k)$ as a function of $a \in \mathbb{A}_k^\times/k^\times$ is, by the known property of a cusp form, rapidly decreasing at ∞ of $\mathbb{A}_k^\times/k^\times$, i.e., when $|a|$ tends to infinity. Therefore, $\int_{\mathbb{A}_k^\times/k^\times} |a|^s F(a, k)d^\times a$ is a convergent integral for $Re(s)$ large enough.

Observe that if the period integral of every function in Π on $Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)$ is zero, then the function $F(a, k)$ will be identically zero too, and hence the period integral $\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)E(g, s)dg$ will be zero at least for $Re(s)$ large, and therefore identically 0 as an analytic function.

On the other hand, as mentioned at the end of proof of Proposition 1 in [AGR] as a well-known fact, $\mathrm{Res}_{s=1} E(g,s)$ is the constant function 1 on $\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)$, we have

$$\mathrm{Res}_{s=1}\left(\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)E(g,s)dg\right) = \int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)dg,$$

a nonzero number by our initial assumption that $\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g)dg \neq 0$ for some $f \in \Pi$, proving by contradiction that the period integral of some function in Π on $Q_n^1(k)\backslash Q_n^1(\mathbb{A}_k)$ must be nonzero. □

Theorem 20. *Let Π be a cuspidal automorphic representation of $\mathrm{U}(W_n \otimes K)$. Then the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$, as well as on the symplectic subgroup $\mathrm{Sp}(W_n)$ is identically zero.*

Proof. We first apply Proposition 130 to conclude that the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ must be identically zero.

Note that the 'boundedness' hypothesis on functions in Π in Proposition 130 is a well-known consequence of cuspidality. The assertion on the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 is a direct consequence of the Proposition 130 by an inductive argument on noting that for both $\mathrm{U}(1,1)$ and $\mathrm{Sp}(2) = \mathrm{SL}(2)$, the Klingen mirabolic subgroup is the group of upper triangular unipotent matrices, and therefore distinction by unipotent group and cuspidality are contradictory to each other. Thus the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ is identically zero.

Now, the theorem follows from Proposition 131. □

Remark 21. The idea of using Eisenstein series in Proposition 131 comes from a reading of [AGR], specially in their Proposition 2, on page 719.

6. Isogenies among classical groups

The rest of the paper uses theta correspondence to classify irreducible admissible representations of $\mathrm{U}_4(F)$ which are distinguished by $\mathrm{Sp}_4(F)$ both locally and globally. To be able to use methods of theta correspondence, we will find it convenient to turn the pair $(\mathrm{U}_4(F), \mathrm{Sp}_4(F))$ into the closely related pair which is $(\mathrm{SO}(4,2), \mathrm{SO}(3,2))$, which we elaborate here for the benefit of some of the readers. Here $\mathrm{SO}(4,2)$ is a special orthogonal group which is not split, but quasi-split and split over the quadratic extension E/F used to define the unitary group $\mathrm{U}(2,2)$, which is also assumed to be quasi-split; the group $\mathrm{SO}(3,2)$ is a split orthogonal group in 5 variables.

6.1. The isogeny $\mathrm{Sp}(4) \to \mathrm{SO}(2,3)$

Let W be a 4 dimensional symplectic space with basis $\{e_1, e_2, e_3, e_4\}$ endowed with the symplectic form

$$A = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

The syplectic group $\mathrm{Sp}(W)$ defined using this symplectic form is also the subgroup of $\mathrm{GL}(W)$ fixing the vector $w_0 = e_1 \wedge e_4 + e_2 \wedge e_3$ in $\bigwedge^2 W$.

Consider the bilinear form $B : \bigwedge^2 W \times \bigwedge^2 W \longrightarrow \bigwedge^4 W \cong F$ given by :

$$(w_1 \wedge w_2, w_3 \wedge w_4) \longrightarrow w_1 \wedge w_2 \wedge w_3 \wedge w_4.$$

It is easily seen that B is a non-degenerate symmetric bilinear form on $\bigwedge^2 W$ on which $g \in \mathrm{GL}(W)$ operates by scaling by $\det g$, i.e., $gB = (\det g)B$, in particular, $\mathrm{SL}(W)$ preserves the bilinear form, giving rise to a homomorphism from $\mathrm{SL}_4(F)$ to the corresponding orthogonal group in 6 variables which is $\mathrm{SO}(3,3)$.

Further,

$$B(w_0, w_0) = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0,$$

hence the orthogonal complement $\langle w_0 \rangle^\perp \subset \bigwedge^2 W$ is a non-degenerate quadratic subspace of $\bigwedge^2 W$ of dimension 5 preserved by $\mathrm{Sp}(W)$.

This gives rise to an isogeny of algebraic groups $\mathrm{Sp}(4) \to \mathrm{SO}(2,3)$, making the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Sp}(4) & \longrightarrow & \mathrm{SL}(4) \\ \big\downarrow & & \big\downarrow \\ \mathrm{SO}(2,3) & \longrightarrow & \mathrm{SO}(3,3). \end{array}$$

6.2. The isogeny $\mathrm{SU}(2,2) \to \mathrm{SO}(4,2)$

In this section we construct an isogeny from $\mathrm{SU}(2,2)$ to $\mathrm{SO}(4,2)$, which although is known to exist by generalities (because both groups are quasi-split over F and split by E, and the first group is simply connected), we have preferred to give an explicit construction in some detail not having found one in the literature (there are some constructions over $\mathbb{R}$). In fact, we were surprised to find that the existence of the isogeny is not there for all hermitian forms (in 4 variables), but only those with discriminant 1, see the remark at the end of the section.

Let E be a quadratic field extension of a field F, with $e \to \bar{e}$ the non-trivial Galois automorphism of E over F. Let V be a vector space over E equipped with a hermitian form $H : V \times V \to E$ such that :

1. $H(v_1 d_1, v_2 d_2) = \bar{d}_1 H(v_1, v_2) d_2$ for all $v_1, v_2 \in V, d_1, d_2 \in E$.
2. $\overline{H(v_1, v_2)} = H(v_2, v_1)$.

Define $\mathrm{U}(V, H)$ to be the corresponding unitary group which is the isometry group of the pair (V, H), and $\mathrm{SU}(V, H)$ to be the subgroup of determinant one E-automorphisms. It will be convenient for us to think of H as a $n \times n$ hermitian matrix over E where $n = \dim V$, which we will actually take to be a symmetric matrix over F, and define $\mathrm{U}(V, H)$ by :

$$\mathrm{U}(V, H) = \{g \in \mathrm{GL}(V) | gH\, {}^t\bar{g} = H.\}$$

Note that $\mathrm{GL}_4(E)$ operates on the space of 4×4 skew-symmetric matrices over E by $g \circ X = gX\, {}^t g$ which carries a quadratic form, the Pfaffian, given on

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & X_{23} & X_{24} \\ -X_{13} & -X_{23} & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix},$$

by (cf. E.Artin's, 'Geometric Algebra', page 142)

$$\mathrm{Pf}(X) = X_{12}X_{34} + X_{13}X_{42} + X_{14}X_{23} = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}.$$

One knows that $\mathrm{Pf}(g \circ X) = \det(g)\mathrm{Pf}(X)$, therefore this gives an explicit homomorphism of $\mathrm{SL}_4(E)$ into $\mathrm{SO}(3, 3)(E)$. In the rest of this section, we will construct a 6 dimensional F-subspace of the space of 4×4 skew-symmetric matrices which is left stable by $\mathrm{SU}(V, H)$, and on which Pfaffian takes values in F giving rise to an isogeny from $\mathrm{SU}(2, 2)$ to $\mathrm{SO}(4, 2)$.

Lemma 132. *There exists an automorphism ϕ of order 2 (well-defined up to ± 1) on the space of 4×4 skew-symmetric matrices over a field F such that,*

$$\phi(gX\, {}^t g) = \det(g)\, {}^t g^{-1} \phi(X) g^{-1}, \qquad (*)$$

for all $g \in \mathrm{GL}_4(F)$ and X any 4×4 skew-symmetric matrix over F ; equivalently, *for a 4 dimensional vector space V over F, we have a natural isomorphism :*

$$\Lambda^2 V \cong \det(V) \otimes \Lambda^2 V^\vee.$$

Further, the automorphism ϕ preserves the Pfaffian : $\mathrm{Pf}(X) = \mathrm{Pf}(\phi(X))$.

Proof. Identifying the space of 4×4 skew-symmetric matrices over the field F to $\Lambda^2 V$, the mapping ϕ is nothing but what's called the Hodge-$\star$ operator (with respect to the quadratic form $X_1^2 + X_2^2 + X_3^2 + X_4^2$) in general from $\Lambda^k V$ to $\Lambda^{n-k} V$; we omit further details. □

Remark 22. One can write down ϕ explicitly as follows :

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & X_{23} & X_{24} \\ -X_{13} & -X_{23} & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix} \longrightarrow \phi(X) = \begin{pmatrix} 0 & X_{34} & -X_{24} & X_{23} \\ -X_{34} & 0 & X_{14} & -X_{13} \\ X_{24} & -X_{14} & 0 & X_{12} \\ -X_{23} & X_{13} & -X_{12} & 0 \end{pmatrix},$$

and it is thus clear too that $\mathrm{Pf}(X) = \mathrm{Pf}(\phi(X))$.

Lemma 133. *Let E be a quadratic separable extension of a field F with $x \to \bar{x}$ the Galois involution of E/F, and let H be any symmetric nonsingular matrix over F with* $\det H = 1$*. Then the automorphism $\phi_H : X \to \phi(H\bar{X}H)$ of the space of 4×4 skew-symmetric matrices over E is of order 2.*

Proof. The square of the automorphism $\phi_H : X \to \phi(H\bar{X}H)$ is the automorphism

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left] & \phi(H\bar{X}H) \arrow[r] & \phi(H\phi(HXH)H) = \det(H)X\ . \end{tikzcd}$$

□

Lemma 134. *The automorphism $X \to \phi(H\bar{X}H)$ on the space of 4×4 skew-symmetric matrices over E commutes with the action of the special unitary group $SU(V, H)$ on this space.*

Proof. We need to prove that :

$$\phi(H\bar{g}\bar{X}\,{}^t\bar{g}H) = g\phi(H\bar{X}H)\,{}^tg,$$

but by the defining property $(*)$ of ϕ (using that H is a nonsingular symmetric matrix over F with $\det H = 1$ and $\det g = 1$), we have

$$\begin{aligned} \phi(H\bar{g}\bar{X}\,{}^t\bar{g}H) &= H^{-1}\,{}^t\bar{g}^{-1}\phi(\bar{X})\bar{g}^{-1}H^{-1}, \\ g\phi(H\bar{X}H)\,{}^tg &= gH^{-1}\phi(\bar{X})H^{-1}\,{}^tg. \end{aligned}$$

Thus if :

$$H^{-1}\,{}^t\bar{g}^{-1}\phi(\bar{X})\bar{g}^{-1}H^{-1} = gH^{-1}\phi(\bar{X})H^{-1}\,{}^tg,$$

we will have proved the lemma. But clearly, this is implied by :

$$H^{-1}\,{}^t\bar{g}^{-1} = gH^{-1},$$

which is equivalent to :

$${}^t\bar{g}Hg = H,$$

which is the definition of the unitary group $\mathrm{U}(V, H)$. $\square$

Note the following general lemma on Galois descent (cf. 'The book of involutions' due to Knus et al, Lemma 18.1, page 279).

Lemma 135. *Let E be a Galois extension of a field F, and W a finite dimensional vector space over E equipped with a semi-linear action of $G = \mathrm{Gal}(E/F)$ on W, i.e., there is an F-linear action $g \to \pi(g)$ of $\mathrm{Gal}(E/F)$ on W with $\pi(g)(ew) = g(e)\pi(g)(w)$ for all $g \in \mathrm{Gal}(E/F)$, $w \in W$. The F-subspace $W_0 = W^G$ of W has the property that $W_0 \otimes E = W$.*

It follows from this lemma that the fixed points of the involution $X \to \phi_H(X) = \phi(H\bar{X}H)$ on the vector space S of skew-symmetric matrices over E is a vector space S_0 over F of dimension 6 with an action of $\mathrm{SU}(V, H)$.

Now

$$q(X) = \mathrm{Pf}(X),$$

the Pfaffian of a skew-symmetric matrix X over E, is an F-valued nondegenerate quadratic form on S_0 which is invariant under $\mathrm{SU}(V, H)$ (since $\mathrm{Pf}(gX\,{}^tg) = \det(g)\mathrm{Pf}(X)$), defining the isogeny $\mathrm{SU}(V, H) \to \mathrm{SO}(S_0)$, which for the unitary group defined by the hermitian form :

$$A = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix},$$

lands inside the orthogonal group $\mathrm{SO}(2, 4)$ which is the orthogonal group of the quadratic form of Witt index 2 over F for $X + E + X^\vee$ where $X, X^\vee$ are maximal isotropic subspaces of W in perfect pairing, and E is a quadratic separable field extension of F with its Norm form.

The isogeny of algebraic groups $\mathrm{Sp}(4) \to \mathrm{SO}(2, 3)$, together with the inclusion of $\mathrm{Sp}(4) \subset \mathrm{SU}(2, 2)$, gives rise to the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Sp}(4) & \longrightarrow & \mathrm{SU}(2,2) \\ \downarrow & & \downarrow \\ \mathrm{SO}(2,3) & \longrightarrow & \mathrm{SO}(2,4). \end{array}$$

Remark 23. The isogeny constructed in this section from $\mathrm{SU}(V, H)$ to an orthogonal group in 6 variables is valid only when one can take $\det H = 1$. For instance, over reals, the group $\mathrm{SU}(3,1)$ cannot be isogenous to any one of the groups $\mathrm{SO}(p,q)$ with $p+q=6$ since an isogeny will also give an isogeny among their maximal compacts, and the maximal compact of $\mathrm{SU}(3,1)$ is $\mathrm{U}(3)$ which is not (isogenous) to the maximal compact subgroup of any one of the $\mathrm{SO}(p,q)$ with $p+q=6$.

7. Weil representation, and its twisted Jacquet modules

Let G be a reductive algebraic group over a non-archimedean local field F, P a parabolic subgroup of G with Levi decomposition $P = MN$, and $\psi : N(F) \longrightarrow \mathbb{C}^\times$ a character on $N(F)$. In analogy with Jacquet modules, one defines the twisted Jacquet module π_ψ, for any smooth representation π of $G(F)$ to be the largest quotient of π on which $N(F)$ operates by $\psi : N(F) \longrightarrow \mathbb{C}^\times$, i.e.,

$$\pi_\psi = \frac{\pi}{\{n \cdot v - \psi(n)v | n \in N(F), v \in \pi\}}.$$

These twisted Jacquet modules define an exact functor from smooth representations of P to smooth representations of $M_\psi(F) = \{m \in M(F) | \psi(mnm^{-1}) = \psi(n), \forall n \in N(F)\}$, i.e., if

$$0 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \pi_3 \longrightarrow 0,$$

is an exact sequence of smooth P-modules, then

$$0 \longrightarrow \pi_{1,\psi} \longrightarrow \pi_{2,\psi} \longrightarrow \pi_{3,\psi} \longrightarrow 0,$$

is an exact sequence of smooth $M_\psi(F)$-modules.

For the dual reductive pair $(\mathrm{O}(V), \mathrm{Sp}(W))$, we will use twisted Jacquet modules of the Weil representation of $\mathrm{Sp}(V \otimes W)$ for P, a Siegel parabolic in $\mathrm{Sp}(W)$, and a character ψ on the unipotent radical of such a parabolic subgroup. The twisted Jacquet module is naturally a representation of $\mathrm{O}(V)$, and its structure allows one to relate theta correspondence to distinction of representations.

Before we recall the result on the twisted Jacquet module of the Weil representation, let's begin by defining the Weil representation itself. Let $W = X \oplus X^\vee$ be a symplectic vector space over a local field F together with its natural symplectic pairing. Given a quadratic space $q : V \to F$, the Weil representation of $\mathrm{Sp}(V \otimes W)$ gives rise to a representation of $\mathrm{O}(V) \times \mathrm{Sp}(W)$ on $\mathbb{S}(V \otimes X^\vee)$, the Schwartz space of locally constant compactly supported functions on $(V \otimes X^\vee)(F)$. The Weil representation depends on the choice of a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ which will be fixed throughout the paper.

Let's note that although one talks of Weil representation of $\mathrm{Sp}(V \otimes W)$, it is in fact a representation of a certain two fold (topological) cover of $\mathrm{Sp}(V \otimes W)$, called the metaplectic cover of $\mathrm{Sp}(V\otimes W)$, and not of $\mathrm{Sp}(V\otimes W)$ itself. If $\dim V$ is even, then this metaplectic cover of $\mathrm{Sp}(V\otimes W)$ splits over $\mathrm{O}(V)\times\mathrm{Sp}(W)$. There is in fact a natural choice of splitting of the metaplectic cover of $\mathrm{Sp}(V\otimes W)$ restricted to $\mathrm{O}(V)\times\mathrm{Sp}(W)$ allowing one to talk of the Weil representation of $\mathrm{O}(V)\times\mathrm{Sp}(W)$ (for $\dim V$ even). In this representation, elements of $\{\phi\in\mathrm{Hom}\,(X^\vee,X)|\phi=\phi^\vee\}\cong\mathrm{Sym}^2X$, which can be identified to the unipotent radical N of the Siegel parabolic in $\mathrm{Sp}(W)$ stabilizing the isotropic subspace X, operate on $\mathbb{S}(V\otimes X^\vee)$ by

$$(n\cdot f)(x)=\psi((q\otimes q_n)x)f(x), \tag{2.1}$$

where $n\in\mathrm{Hom}\,(X^\vee,X)$ gives rise to a quadratic form $q_n:X^\vee\to F$, which together with the quadratic form $q:V\to F$, gives rise to the quadratic form $q\otimes q_n:V\otimes X^\vee\to F$ defined by $(q\otimes q_n)(v\otimes w')=q(v)\cdot q_n(w')$.

The Weil representation realized on $\mathbb{S}(V\otimes X^\vee)$ has the natural action of $\mathrm{O}(V)$ operating as

$$L(h)\varphi(x)=\varphi(h^{-1}x).$$

The group $\mathrm{GL}(X)$ sits naturally inside $\mathrm{Sp}(X\oplus X^\vee)$ (preserving X and $X^\vee$), and its action on $\mathbb{S}(V\otimes X^\vee)$ is given by

$$L(g)\varphi(x)=\chi_V(\det g)|\det g|^{m/2}\varphi(gx),$$

where $m=\dim V$, χ_V is the quadratic character of $F^\times$ given in terms of the Hilbert symbol as $\chi_V(a)=(a,\mathrm{disc}V)$ with $\mathrm{disc}V$ the normalized discriminant of V. These actions together with the action of the Weyl group element (which acts on $\mathrm{GL}(X)$ sitting inside $\mathrm{Sp}(X\oplus X^\vee)$ through $A\to {}^tA^{-1}$) of $\mathrm{Sp}(W)$ through Fourier transforms on $\mathbb{S}(V\otimes X^\vee)$ — but which we will not define precisely, gives the action of $\mathrm{O}(V)\times\mathrm{Sp}(W)$ on $\mathbb{S}(V\otimes X^\vee)$.

The Weil representation thus gives rise to a representation of the group $\mathrm{O}(V)\times\mathrm{Sp}(W)$. Given an irreducible representation π of $\mathrm{O}(V)$, there exists a representation $\Theta(\pi)$ of $\mathrm{Sp}(W)$ of finite length, such that $\pi\otimes\Theta(\pi)$ is the maximal π-isotypic quotient of ω. It was conjectured by R. Howe that the representation $\Theta(\pi)$ of $\mathrm{Sp}(W)$ has a unique irreducible quotient $\theta(\pi)$; this conjecture which was proved by Howe in the archimedean case, by Waldspurger in the non-archimedean case for odd residue characteristic, is now proved in complete generality by W-T. Gan and S. Takeda, cf. [GT]. When one talks about the theta correspondence, one means the correspondence $\pi\to\theta(\pi)$. One can reverse the roles of the groups $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ and begin with an irreducible representation π of $\mathrm{Sp}(W)$, and define a representation $\Theta(\pi)$ of $\mathrm{O}(V)$ of finite length, and also the unique irreducible quotient $\theta(\pi)$.

Since N, the unipotent radical of the Siegel parabolic of $\mathrm{Sp}(W)$ is a finite dimensional vector space over F isomorphic to the space of symmetric elements in $\mathrm{Hom}\,[X^\vee,X]$, i.e., $\phi\in\mathrm{Hom}\,[X^\vee,X]$ such that $\phi^\vee=\phi$, as discussed in the section on Notation, one can identify the space of characters $\lambda:N\to\mathbb{C}^\times$ to symmetric elements in $\mathrm{Hom}\,[X, X^\vee]$, i.e., to quadratic forms on X, through the natural non-degenerate pairing :

$$\mathrm{Hom}\,(X^\vee, X) \times \mathrm{Hom}\,(X, X^\vee) \longrightarrow \mathrm{Hom}\,(X^\vee, X^\vee) \xrightarrow{\mathrm{tr}} F.$$

Now given a linear map $x : X \to V$, one can restrict a quadratic form on V to one on X ; this construction plays an important role in the following well-known proposition for which we refer to [PR], Corollary 6.2.

Lemma 136. *The twisted Jacquet module of the Weil representation corresponding to the dual reductive pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ for N, the unipotent radical of the Siegel parabolic in $\mathrm{Sp}(W)$ stabilizing $X \subset W$, a maximal isotropic subspace in W, is nonzero exactly for those characters of N which correspond to the 'restriction' of quadratic form on V to X via a linear map $x : X \to V$.*

Proposition 137. *The twisted Jacquet module of the Weil representation of the dual reductive pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ for N, the unipotent radical of the Siegel parabolic in $\mathrm{Sp}(W)$ stabilizing $X \subset W$, a maximal isotropic subspace in W, for the characters of N which corresponds to a non-degenerate quadratic form on X, which we assume is obtained by restriction of the quadratic form on V via a linear map $x : X \to V$ is as a representation of $\mathrm{O}(V)$ the representation*

$$\mathrm{ind}_{\mathrm{O}(X^\perp)}^{\mathrm{O}(V)} \mathbb{C},$$

where $\mathrm{O}(X^\perp)$ is the orthogonal group of the orthogonal complement of X inside V sitting inside $\mathrm{O}(V)$ by acting trivially on X.

Remark 24. Assuming that $\mathrm{Sp}(W) = \mathrm{SL}_2(F)$, so that $\dim X = 1$, in which case the previous proposition identifies irreducible representations π of $\mathrm{O}(V)$ which are distinguished by $\mathrm{O}(X^\perp)$ to theta lifts of (suitable) representations of $\mathrm{SL}_2(F)$. Observe that if π remains irreducible when restricted to $\mathrm{SO}(V)$, therefore the representations π and $\pi \otimes \det$ of $\mathrm{O}(V)$ are distinct, π restricted to $\mathrm{SO}(V)$ is distinguished by $\mathrm{SO}(X^\perp)$ if and only if one of the representations π or $\pi \otimes \det$ of $\mathrm{O}(V)$ is distinguished by $\mathrm{O}(X^\perp)$ if and only if one of the representations π or $\pi \otimes \det$ of $\mathrm{O}(V)$ arises as a theta lift from (a suitable representation of) $\mathrm{SL}_2(F)$.

Remark 25. There are what are called *conservation relations*, now proved in all generality in [SZ], which for a representation π of $\mathrm{O}(V)$ dictate a relationship between first occurrence of π in the tower with members $\mathrm{Sp}_{2n}(F)$, with the first occurrence of $\pi \otimes \det$ in the same tower. If we are dealing with representations π of $\mathrm{O}(V)$, $\dim V \geq 4$, arising from theta correspondence with $\mathrm{SL}_2(F)$, these conservation relations will force the first occurrence of $\pi \otimes \det$ to be much later. As a result, π cannot be isomorphic to $\pi \otimes \det$, equivalently, π restricted from $\mathrm{O}(V)$ to $\mathrm{SO}(V)$ must remain irreducible. Thus it is legitimate for us to use theta correspondence between $\mathrm{SL}_2(F)$ and $\mathrm{SO}(V)$ instead of $\mathrm{SL}_2(F)$ and $\mathrm{O}(V)$.

Corollary 4. *Assume that* $\mathrm{Sp}(W) = \mathrm{SL}_2(F)$*, so that* $\dim X = 1$*. Embed* X *as a one-dimensional non-degenerate subspace of* V*, such that as a quadratic space* X *is isomorphic to the quadratic space* ax^2 *for* $a \in F^\times$*. Then for an irreducible admissible representation* μ *of* $\mathrm{SO}(V)$ *which is distinguished by* $\mathrm{SO}(X^\perp)$ *its big theta lift* $\Theta(\mu)$ *to* $\mathrm{SL}_2(F)$ *is a representation of* $\mathrm{SL}_2(F)$ *which has a Whittaker model for the character* $\psi_a(x) = \psi(ax)$ *(in particular,* $\theta(\mu) \neq 0$*, although because of the difference between* $\Theta(\mu)$ *and* $\theta(\mu)$*,* $\theta(\mu)$ *may not have a Whittaker model for the character* $\psi_a(x) = \psi(ax)$*); conversely, if an irreducible admissible representation of* $\mathrm{SO}(V)$ *is obtained as (small) theta lift* $\theta(\pi)$ *of an irreducible admissible representation* π *of* $\mathrm{SL}_2(F)$ *which has a Whittaker model for the character* $\psi_a(x) = \psi(ax)$*, then* $\theta(\pi)$ *is distinguished by* $\mathrm{SO}(X^\perp)$*.*

Remark 26. It should be emphasized that in the corollary, we take small theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V)$, but big theta lift from $\mathrm{SO}(V)$ to $\mathrm{SL}_2(F)$. It is known that the various sub-quotients of the representation $\Theta(\mu)$ on $\mathrm{SL}_2(F)$ have the same cuspidal support, and therefore if $\theta(\mu)$ is either cuspidal, or is an irreducible principal series, we can replace $\Theta(\mu)$ in the corollary by $\theta(\mu)$. However, if $\theta(\mu)$ is a component of a reducible principal series, there is a definite possibility of having a difference between $\Theta(\mu)$ and $\theta(\mu)$ which can affect the conclusion of the corollary (if we were to replace $\Theta(\mu)$ by $\theta(\mu)$).

Remark 27. A consequence of the above corollary is that small theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V)$, V any quadratic space of dimension $n \geq 4$, of different irreducible (infinite dimensional) representations of $\mathrm{SL}_2(F)$ which belong to the *same* L-packet, and therefore have Whittaker model for characters $\psi_a(x) = \psi(ax)$, for which $a \in F^\times/F^{\times 2}$ belong to *different* cosets, are distinguished by $\mathrm{SO}(X_a^\perp)$ where $X_a^\perp$ is the orthogonal complement of the quadratic subspace ax^2 of V; these subspaces $X_a^\perp$ have different discriminants, and therefore belong to different pure innerforms of $\mathrm{SO}_{n-1}(F)$. Thus assuming that the theta lift of an L-packet on $\mathrm{O}(V)$ to $\mathrm{SL}_2(F)$ makes up a subset of an L-packet on $\mathrm{SL}_2(F)$, we are able to make a contribution to the Gan-Gross-Prasad conjectures for non-tempered representations : that inside an L-packet on $\mathrm{SO}(V)$, there is a unique member which is distinguished by $\mathrm{SO}(W)$ for W a fixed codimension one subspace of V, i.e., multiplicity one holds in such an L-packet (and these representations on $\mathrm{SO}(V)$ arise by theta lift from $\mathrm{SL}_2(F)$); further, if instead of V we take the unique other quadratic space V' over F with the same discriminant as V, then for $W' = X_a'^{\perp}$, the orthogonal complement of ax^2 in V', the same analysis proves that a theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V)$ is distinguished by $\mathrm{SO}(W)$ if and only if the theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V')$ is distinguished by $\mathrm{SO}(W')$, i.e., in the extended Vogan L-packet of the pair $(\mathrm{SO}(V), \mathrm{SO}(W))$, the multilicity of distinguished representations is 2 instead of 1 in the usual Gross-Prasad conjectures (for generic L-packets).

Remark 28. Corollary 4 in various forms has been around in the literature, for example let's briefly compare it to the work of Waldspurger [Wa] on toric periods, see e.g., Proposition 14 in [Wa]. In this work of Waldspurger, which is for V a quadratic space of dimension 3, in which case $\mathrm{SO}(V)$ is either $\mathrm{PGL}_2(F)$ or $\mathrm{PD}^\times$, for the unique quaternion division algebra D over F, and $\mathrm{SO}(X^\perp)$ is $E^\times/F^\times$ where E is a quadratic algebra over F with the natural embeddings $E^\times/F^\times \hookrightarrow \mathrm{PGL}_2(F)$, and $E^\times/F^\times \hookrightarrow \mathrm{PD}^\times$. Since $\dim(V) = 3$, Waldspurger deals with the metaplectic cover $\overline{\mathrm{SL}}_2(F)$ of $\mathrm{SL}_2(F)$, and concludes as we do, that there is a bijective correspondence between representations of $\mathrm{PGL}_2(F)$ and $\mathrm{PD}^\times$ which have a nontrivial toric period for $E^\times/F^\times$ with the corresponding representations of $\overline{\mathrm{SL}}_2(F)$ which have a nontrivial Whittaker functional. For $\dim(V) = 4$, compare corollary 4 to the work of Brooks Roberts [Ro], Theorem 7.4 and corollary 7.5.

8. A lemma on twisted Jacquet modules

The aim of this section is to fill in a certain detail in Lemma 6.3 of [PT]. For this purpose we first recall that lemma (in a suitably modified form).

Lemma 138. *Let X be the F-rational points of an algebraic variety defined over a local field F. Let P be a locally compact totally disconnected group with $P = MN$ for a normal subgroup N of P which we assume is a union of compact subgroups. Assume that P operates smoothly on $\mathbb{S}(X)$, and that the action of P restricted to M is given by an action of M on X. Suppose that there is a continuous map from X to characters on $N(F)$, $x \to \psi_x$, such that N operates on $\mathbb{S}(X)$ by $(n \cdot f)(x) = \psi_x(n)f(x)$. Fix a character $\psi : N \to \mathbb{C}^\times$, and let M_ψ denote the subgroup of M which stabilizes the character ψ of N. The group M_ψ acts on the set of points $x \in X$ such that $\psi_x = \psi$. Denote this set of points in X by X_ψ which we assume to be closed in X. Then,*

$$\mathbb{S}(X)_\psi \cong \mathbb{S}(X_\psi)$$

as M_ψ-modules.

The proof of this lemma in [PT] depends on the exact sequence of M_ψ-modules,

$$0 \longrightarrow \mathbb{S}(X - X_\psi) \longrightarrow \mathbb{S}(X) \longrightarrow \mathbb{S}(X_\psi) \longrightarrow 0.$$

It is asserted in [PT] that since taking the ψ-twisted Jacquet functor is exact, and $\mathbb{S}(X - X_\psi)_\psi = 0$, the lemma follows. However, the fact that $\mathbb{S}(X - X_\psi)_\psi = 0$, needs an argument which we supply now.

Lemma 139. *With the same conditions as in Lemma 138, assume that ψ is a character of N which is not of the form ψ_x for any $x \in X$, then the twisted Jacquet module $\mathbb{S}(X)_\psi = 0$.*

Proof. By twisting the action of N on $\mathbb{S}(X)$ by ψ^{-1}, it suffices to assume that $\psi = 1$, so that we are dealing with standard Jacquet modules.

Since N operates on $\mathbb{S}(X)$ by $(n \cdot f)(x) = \psi_x(n)f(x)$, it is clear that N leaves $\mathbb{S}(X')$ invariant for any X' which is a compact open subset of X. Since X is a union of compact open subsets, $\mathbb{S}(X)$ is a union (direct limit) of $\mathbb{S}(X')$ where X' runs over all compact open subsets of X. It is easy to see that to prove that the Jacquet module $\mathbb{S}(X)_N = 0$, it suffices to prove that $\mathbb{S}(X')_N = 0$ for any compact open subset X' of X.

To prove that $\mathbb{S}(X')_N = 0$, we need to prove that

$$\begin{aligned}\mathbb{S}(X')[N] &:= \{f - n \cdot f | n \in N(F), f \in \mathbb{S}(X')\} \\ &= \{(1 - \psi_x(n))f(x) | n \in N(F), f \in \mathbb{S}(X')\} \\ &= \mathbb{S}(X').\end{aligned}$$

It is clear that the subspace of $\mathbb{S}(X')$ generated by functions of the form $(1 - \psi_x(n))f(x)$ where $n \in N(F)$, and $f \in \mathbb{S}(X')$ is an ideal in $\mathbb{S}(X')$. If this was a proper ideal, it would be contained in a maximal ideal, and therefore by the well-known Gelfand-Naimark theorem, all functions in this subspace must vanish at some point $x_0 \in X'$. (We took X' to be compact to be able to apply Gelfand-Naimark theorem ; also it may be mentioned that although $\mathbb{S}(X')$ is not the space of *all* continuous functions on X', the conclusion of Gelfand-Naimark theorem — and its proof — that the maximal ideals in the space of continuous functions $\mathbb{C}(X')$ are in bijective correspondence with points of X' is the same for $\mathbb{S}(X')$.)

For the space of functions generated by $(1 - \psi_x(n))f(x)$ where $n \in N(F)$, and $f \in \mathbb{S}(X')$, to vanish at $x_0 \in X'$, we must have $(1 - \psi_{x_0}(n)) = 0$ for all $n \in N(F)$, which is the same as saying $\psi_{x_0} = 1$, a contradiction to our hypothesis that the character ψ (taken to be trivial) is not among the characters $\psi_x, x \in X$, proving that $\mathbb{S}(X)_\psi = 0$. $\square$

9. Application to distinction of representations

Let $V = X + E + X^\vee$ be a quadratic space of dimension 6 where X and $X^\vee$ are totally isotropic subspaces of V of dimension 2 over F in duality with each other under the associated bilinear form, and both perpendicular to the space E which is a quadratic field extension of F with its associated norm form $\mathbb{N}m(e) = e\bar{e}$. Thus the orthogonal group $\mathrm{SO}(V)$ is a quasi-split orthogonal group which is split by E, and may be written as $\mathrm{SO}(4, 2)$. It is clear that the one dimensional quadratic space aX^2 can be embedded inside $(E, \mathbb{N}m)$ as a quadratic subspace if and only if $a \in F^\times$ is a norm from $E^\times$.

Since V is an isotropic quadratic space, it represents all elements of $F^\times$, i.e., given $a \in F^\times$, there exists $v \in V$ such that $q(v) = a$. It follows that for the corresponding quadratic subspace $X \subset V$, $X^\perp$ is a split quadratic space if and only if $a \in F^\times$ is a norm from $E^\times$, in which case $\mathrm{SO}(X^\perp)$ could be written as $\mathrm{SO}(3, 2)$; if $a \in F^\times$ is not a norm from $E^\times$, $\mathrm{SO}(X^\perp)$ could be written as $\mathrm{SO}(4, 1)$

as it is then a quasi-split form of $\mathrm{SO}(5)$ of rank 1 which is split by E.

Proposition 140. *Let π be an irreducible admissible representation of $\mathrm{SL}_2(F)$ which is obtained as a theta lift from $\mathrm{O}(2) = \mathrm{O}(E)$. Then if π has a Whittaker model for the characters $\psi_a(x) = \psi(ax)$ then a must belong to $\mathbb{N}m(E^\times)$. Conversely, an irreducible admissible representation of $\mathrm{SL}_2(F)$ which is dihedral with respect to E, i.e., is obtained as a theta lift from $\mathrm{O}(b \cdot E)$ for some $b \in F^\times$, and has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for $a \in \mathbb{N}m(E^\times)$, then it is obtained as a theta lift from $\mathrm{O}(2) = \mathrm{O}(E)$.*

Proof. From equation (1),

$$(n \cdot f)(x) = \psi((q \otimes q_n)x)f(x),$$

with $x \in E$, $(q \otimes q_n)(x) = n^2\mathbb{N}m(x)$, so the first part of the proposition follows. For the second part of the proposition, observe that by the first part of the proposition, if a representation of $\mathrm{SL}_2(F)$ is obtained as a theta lift of a representation of $\mathrm{O}(b \cdot E)$, then it has Whittaker model only for characters of the form $\psi_{bc}(x) = \psi(bcx)$ for some $c \in \mathbb{N}mE^\times$. Since it is given that π has a Whittaker model for $a \in \mathbb{N}mE^\times$, it follows that $b \in \mathbb{N}mE^\times$. Since $b \in \mathbb{N}mE^\times$, it follows that $b \cdot E \cong E$ as quadratic spaces, and hence π is indeed obtained as theta lift from $\mathrm{O}(E)$ as desired. $\square$

Proposition 141. *For an irreducible admissible representation π of $\mathrm{SL}_2(F)$, the following are equivalent :*

1. *π has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$.*
2. *If π is obtained as a theta lift of a representation of $\mathrm{O}(2) = \mathrm{O}(b \cdot E)$ for some $b \in F^\times$, it is obtained as a theta lift from $\mathrm{O}(E)$; equivalently, $b \in \mathbb{N}E^\times$ so that $b \cdot E \cong E$ as quadratic spaces.*

Proof. We give a proof by a case-by-case analysis.

1. The representation π is contained in an irreducible representation $\tilde{\pi}$ of $\mathrm{GL}_2(F)$ which remains irreducible when restricted to $\mathrm{SL}_2(F)$, i.e., is π, when restricted to $\mathrm{SL}_2(F)$. In this case, π and $\tilde{\pi}$ have Whittaker model for *all* (non-trivial) characters of F, so nothing to be done in this case, i.e., (1) is true, and (2) is vacuously true.
2. The representation π is contained in an irreducible representation $\tilde{\pi}$ of $\mathrm{GL}_2(F)$ which decomposes into 2 or 4 components when restricted to $\mathrm{SL}_2(F)$, but $\tilde{\pi}$ does not arise from a character of $E^\times$. Let L be the compositum of all quadratic extensions M of F such that $\tilde{\pi}$ is a dihedral representation corresponding to a character of $M^\times$. Then L is either a quadratic or bi-quadratic extension of F such that π has Whittaker model exactly for those characters of the form $\psi_a(x) = \psi(ax)$ for a belonging to a fixed coset of $F^\times/\mathbb{N}m(L^\times)$. It is easy to see that since E is not contained in L, such a coset must intersect $\mathbb{N}m(E^\times)$, i.e., the map :
$$\mathbb{N}m(E^\times) \longrightarrow F^\times/\mathbb{N}mL^\times,$$

must be surjective, i.e., $F^\times = \mathbb{N}mE^\times \cdot \mathbb{N}mL^\times$. But $\mathbb{N}mE^\times$, is a subgroup of $F^\times$ of index 2, therefore $F^\times = \mathbb{N}mE^\times \cdot \mathbb{N}mL^\times$ if and only if,

$$\mathbb{N}mL^\times \not\subset \mathbb{N}mE^\times.$$

But by classfield theory,

$$\mathbb{N}mL^\times \subset \mathbb{N}mE^\times \iff E^\times \subset L^\times.$$

By hypothesis, in this case $E^\times \not\subset L^\times$, so the map : $\mathbb{N}m(E^\times) \longrightarrow F^\times/\mathbb{N}mL^\times$ is surjective.

It follows that in this case π always has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$, and (2) is vacuously satisfied.

3. The representation π is obtained as a theta lift from $\mathrm{O}(b \cdot E)$ for *some* $b \in F^\times$. In this case, the conclusion is part of the previous proposition.

□

Theorem 29. *An irreducible admissible representation of* $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ *is distinguished by* $\mathrm{SO}(3, 2)$ *if and only if it is obtained as a theta lift of a representation* π *of* $\mathrm{SL}_2(F)$ *which has either of the following equivalent properties :*

1. *π has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for $a \in \mathbb{N}m(E^\times)$.*
2. *If π is obtained as a theta lift of a representation of $\mathrm{O}(2) = \mathrm{O}(b \cdot E)$ for some $b \in F^\times$, it is obtained as a theta lift from $\mathrm{O}(E)$.*

Proof. By Corollary 4, we already know that an irreducible admissible representation of $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ is distinguished by $\mathrm{SO}(3, 2)$ if and only if it is obtained as a theta lift of a representation π of $\mathrm{SL}_2(F)$ which has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$. (Observe that by the theorem on 'stable range', since the split rank of $\mathrm{SO}(4, 2)$ is 2, every irreducible admissible representation of $\mathrm{SL}_2(F)$ has a nonzero theta lift to $\mathrm{SO}(4, 2)$.)

Equivalence of (1) and (2) is the content of the previous proposition. □

Corollary 5. *An irreducible admissible supercuspidal representation of* $\mathrm{SO}(X+E+X^\vee) = \mathrm{SO}(4, 2)$ *cannot be distinguished by* $\mathrm{SO}(3, 2)$*. A supercuspidal representation of* $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ *which is obtained as a theta lift from* $\mathrm{SL}_2(F)$ *is distinguished by* $\mathrm{SO}(4, 1)$*.*

Proof. To prove the corollary it suffices to note that by theorem 1, a supercuspidal representation of $\mathrm{SO}(4, 2)$ distinguished by $\mathrm{SO}(3, 2)$ must be obtained as a theta lift of a representation of $\mathrm{SL}_2(F)$ which has a Whittaker model for the character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$.

By Proposition 3, a representation π of $\mathrm{SL}_2(F)$ which has a Whittaker model for the character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$ is either

1. obtained as a theta lift from $\mathrm{O}(2) = \mathrm{O}(E)$, and therefore by the Kudla's theory of towers of theta lifts, the theta lift of such a representation of $\mathrm{SL}_2(F)$ to $\mathrm{O}(X + E + X^\vee) = \mathrm{O}(4, 2)$ cannot be supercuspidal, or
2. the representation π is not obtained as a theta lift from $\mathrm{O}(bE)$ for any $b \in F^\times$. In this case, the first occurrence of π in the tower $\mathrm{O}(V_{b,r}) = \mathrm{O}(Y_r + bE + Y_r^\vee)$, where Y_r has dimension r and $b \in F^\times$, and hence $V_{b,r}$ has dimension $2 + 2r$, has $\dim(V_{b,r}) \geq 4$ for any $b \in F^\times$. Since the sum of the first occurrences in the two towers is 8 by the 'conservation relations', π lifts to both the towers for $\dim(V_{b,r}) = 4$, in particular π lifts to $\mathrm{O}(Y_r + E + Y_r^\vee)$ for $\dim Y_r = 1$, i.e., to $\mathrm{O}(3, 1)$. Again, the lift of π to $\mathrm{O}(4, 2)$ cannot be supercuspidal.

For the second assertion contained in the corollary regarding distinction by $\mathrm{SO}(4, 1)$, note that by the previous analysis, the only supercuspidal representation of $\mathrm{SO}(4, 2)$ which is obtained as a theta lift from a representation π of $\mathrm{SL}_2(F)$ has the property that π is obtained as a theta lift from $\mathrm{O}(b \cdot E)$ for $b \in F^\times - \mathbb{N}mE^\times$. By the conservation relations, theta lift of such representations to $\mathrm{SO}(4, 2)$ are indeed supercuspidal (being the first occurrence), and by Corollary 4, these representations of $\mathrm{SO}(4, 2)$ are distinguished by $\mathrm{SO}(W)$, where W is the orthogonal complement of $b \cdot E$ inside the quadratic space $X + E + X^\vee$ with $\dim X = 2$ (it is easily seen that $b \cdot E$ is contained in the quadratic space $X + E + X^\vee$). Such a W can be seen to be the unique non-split quadratic space of dimension 5 with trivial discriminant, thus $\mathrm{SO}(W) = \mathrm{SO}(4, 1)$. □

We will not go into any details of the corresponding global theorem except to state the following theorem which is a simple consequence of Theorem 11 of [PT].

Theorem 30. *For a cuspidal automorphic representation π of $\mathrm{SL}_2(\mathbb{A}_k)$ which has a Whittaker model for a character $\psi_{0,a}(x) = \psi_0(ax)$ for $a \in \mathbb{N}m(K^\times)$, its theta lift $\Theta(\pi)$ to $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)(\mathbb{A}_k)$ has convergent, and nonzero period integral on $\mathrm{SO}(3, 2)(k)\backslash\mathrm{SO}(3, 2)(\mathbb{A}_k)$. Conversely, if a cuspidal automorphic representation of $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ has nonzero period integral on $\mathrm{SO}(3, 2)(k)\backslash\mathrm{SO}(3, 2)(\mathbb{A}_k)$, it is obtained as a theta lift of a cuspidal automorphic representation π of $\mathrm{SL}_2(\mathbb{A}_k)$ which has a Whittaker model for a character $\psi_{0,a}(x) = \psi_0(ax)$ for $a \in \mathbb{N}m(K^\times)$.*

10. Interpretation via Langlands parameters

We begin with the following most natural conjecture regarding distinction of representations of unitary groups by the symplectic group, for which we indicate a proof for the case of $\mathrm{U}(2, 2)$ dealt with in this paper.

Conjecture 2. *For F a local field, let $\{\pi\}$ be an L-packet of irreducible admissible representations of $\mathrm{U}(n, n)(F)$ which we assume to be the L-packet associated to an*

Arthur packet on $\mathrm{U}(n,n)(F)$*. Then some member of the set* $\{\pi\}$ *is distinguished by* $\mathrm{Sp}_{2n}(F)$ *if and only if under basechange, the representation* $BC(\pi)$ *of* $\mathrm{GL}_{2n}(E)$ *is distinguished by* $\mathrm{Sp}_{2n}(E)$*.*

Remark 31. Given the classification of representations of $\mathrm{GL}_{2n}(E)$ which are distinguished by $\mathrm{Sp}_{2n}(E)$ — which we will recall below — a consequence of the above conjecture is that there should be no tempered representations of $\mathrm{U}(n,n)(F)$ which are distinguished by $\mathrm{Sp}_{2n}(F)$. Recall that in an earlier section, we have proved that there are no cuspidal representations of $\mathrm{U}(n,n)(F)$ which are distinguished by $\mathrm{Sp}_{2n}(F)$.

We next recall the theorem of Offen-Sayag about symplectic periods of representations on $\mathrm{GL}_{2n}(F)$ in terms of Langlands parameters.

Let $W'_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group of F. Let $W''_F = W'_F \times \mathrm{SL}_2(\mathbb{C}) = W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. There is a natural homomorphism $\iota : W'_F \to W''_F = W'_F \times \mathrm{SL}_2(\mathbb{C})$ in which the mapping from W'_F to itself is the identity map, and the mapping from $W'_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$ is trivial on $\mathrm{SL}_2(\mathbb{C})$, and on W_F is given by

$$w \longrightarrow \begin{pmatrix} \nu^{1/2} & 0 \\ 0 & \nu^{-1/2} \end{pmatrix},$$

where ν is the character of W_F (thus factoring through $F^\times$) which is unramified, and takes a uniformizer in $F^\times$ to q^{-1} where q is the cardinality of the residue field.

The mapping $\iota : W'_F \to W''_F = W'_F \times \mathrm{SL}_2(\mathbb{C})$ allows one to restrict admissible homomorphisms of W''_F to $\mathrm{GL}_m(\mathbb{C})$ (whose restriction to W_F have bounded image) to admissible homomorphisms of W'_F to $\mathrm{GL}_m(\mathbb{C})$ which are certain Langlands parameters of irreducible admissible unitary representations of $\mathrm{GL}_m(F)$. Admissible representations of W''_F (whose restriction to W_F have bounded image) are called Arthur parameters, and their restriction to W'_F via the mapping $\iota : W'_F \to W''_F$ is called the Langlands parameter associated to an Arthur parameter. (By the work of Moeglin-Waldspurger, such Langlands parameters account for all representations of $\mathrm{GL}_m(F)$ which arise in the theory of automorphic forms.)

Let St_n denote the unique irreducible $\mathbb{C}$-representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n.

Theorem 32. *(Offen-Sayag) Let* π *be the irreducible admissible unitary representation of* $\mathrm{GL}_{2n}(F)$ *with Langlands parameter* $\sigma_\pi \circ \iota : W'_F \to \mathrm{GL}_{2n}(\mathbb{C})$ *for an admissible representations* σ_π *of* $W''_F = W'_F \times \mathrm{SL}_2(\mathbb{C})$ *of dimension* $2n$ *written in the form :*

$$\sigma_\pi = \sum_i \sigma_i \otimes \mathrm{St}_i,$$

where σ_i *are admissible (bounded) representations of* W'_F*. Then the representation* π *has a symplectic model if and only if* $\sigma_i = 0$ *for* i *an odd integer.*

It follows that the Langlands parameters of representations of $\mathrm{GL}_{2n}(F)$ with symplectic period have the shape :

$$\sigma_\pi = \sum_i \sigma_i \otimes [\nu^{(2i-1)/2} + \nu^{(2i-3)/2} + \cdots + \nu^{-(2i-3)/2} + \nu^{-(2i-1)/2}],$$

where σ_i are 'tempered' parameters of W_F'.

Suppose now that we are considering representations of $\mathrm{GL}_{2n}(E)$ with symplectic period which arise by basechange from representations of $\mathrm{U}(n,n)(F)$. The Langlands parameter of such representations are conjugate-selfdual, and therefore in the decomposition :

$$\sigma_\pi = \sum_i \sigma_i \otimes [\nu^{(2i-1)/2} + \nu^{(2i-3)/2} + \cdots + \nu^{-(2i-3)/2} + \nu^{-(2i-1)/2}],$$

the representations σ_i of W_E' are also conjugate-selfdual.

By the calculation done in [GGP], the component group of such parameters of $\mathrm{U}(n,n)(F)$ are trivial, i.e., the L-packet of such representations of $\mathrm{U}(n,n)(F)$ consists of single elements (because of the presence of non-trivial powers of ν in $\sigma_i \otimes \nu^{j/2}$ which appear in σ_π, none of these can be conjugate-selfdual). We note this as a proposition.

Proposition 142. *For F a local field, let $\{\pi\}$ be an L-packet of irreducible admissible representations of $\mathrm{U}(n,n)(F)$ which we assume to be the L-packet associated to an Arthur packet on $\mathrm{U}(n,n)(F)$. Then if under basechange, the representation $BC(\pi)$ of $\mathrm{GL}_{2n}(E)$ is distinguished by $\mathrm{Sp}_{2n}(E)$, the L-packet $\{\pi\}$ must consist of a single member.*

In the rest of this section, we indicate how our work in this paper is in conformity with Conjecture 1 in the case of $\mathrm{U}(2,2)$.

Recall that the L-group of the quasi-split group $\mathrm{SO}(4,2)$ over F which is split by the quadratic extension E of F can be taken to be $\mathrm{O}(6,\mathbb{C})$, such that a Langlands parameter for $\mathrm{SO}(4,2)$ consists of an admissible homomorphism $\sigma : W_F' \longrightarrow \mathrm{O}(6,\mathbb{C})$ with $\det \sigma = \omega_{E/F}$, the quadratic character of $F^\times$ associated by classfield theory to the extension E/F.

It follows by the formalism of theta lifts that if the Langlands parameter of the representation π of $\mathrm{SL}_2(F)$ is $\sigma_\pi : W_F' \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{SO}(3,\mathbb{C})$, then in Theorem 32, the Langlands parameter of the representation $\theta(\pi)$ of $\mathrm{SO}(4,2)$ is the following representation of W_F' :

$$\omega_{E/F}\sigma_\pi + \mathrm{St}_3, \tag{2.2}$$

where we have denoted by St_3 the 3-dimensional representation of W_F which is $[\nu^{-1} + 1 + \nu]$ (thus the present St_3 is what would be denoted earlier by $\mathrm{St}_3 \circ \iota$).

On the other hand, for a conjugate-symplectic parameter $\lambda : W_E' \to \mathrm{GL}_4(\mathbb{C})$, arising from a representation of $\mathrm{U}_4(F)$, $\det(\lambda)^{-1/2}\Lambda^2(\lambda)$ is a 6-dimensional representation with values in $\mathrm{O}_6(\mathbb{C})$, where $\det(\lambda)^{1/2}$ is a character of W_E whose square is $\det(\lambda)$, and the square root must exist if the representation of $\mathrm{U}_4(F)$ can be related to one of $\mathrm{SO}_6(F)$ (since there is a homomorphism from $\mathrm{SU}_4(F)$ to $\mathrm{SO}_6(F)$ with kernel $\pm 1 \subset \mathrm{SU}_4(F)$, only those representations of $\mathrm{SU}_4(F)$ descend to representations of $\mathrm{SO}_6(F)$ which are trivial on $\pm 1 \subset \mathrm{SU}_4(F)$).

Note that if $\lambda = \sigma \otimes \mathrm{St}_2$ is a conjugate-symplectic representation of W'_E (the only non-trivial option allowed by the theorem of Offen-Sayag which we are applying after basechanging the representation of $\mathrm{U}(2,2)(F)$ to $\mathrm{GL}_4(E)$), then,

$$\Lambda^2(\sigma \otimes \mathrm{St}_2) = \Lambda^2(\sigma) \otimes \mathrm{Sym}^2(\mathrm{St}_2) + \mathrm{Sym}^2(\sigma) \otimes \Lambda^2(\mathrm{St}_2) = \det(\sigma)\mathrm{St}_3 + \mathrm{Sym}^2(\sigma).$$

Since $\det(\lambda) = \det(\sigma \otimes \mathrm{St}_2) = \det(\sigma)^2$, we can take $\det(\lambda)^{1/2} = \det(\sigma)$, and hence,

$$\det(\lambda)^{-1/2}\Lambda^2(\sigma \otimes \mathrm{St}_2) = \mathrm{St}_3 + (\det \sigma)^{-1}\mathrm{Sym}^2(\sigma). \tag{2.3}$$

Since $\lambda = \sigma \otimes \mathrm{St}_2$ is a conjugate-symplectic representation of W'_E, σ must be a conjugate-orthogonal representation of W'_E which by Proposition 6.1 of [GGP2] arises (up to a twist by a character of $E^\times$) as basechange of a representation of W'_F and the representation $(\det \sigma)^{-1}\mathrm{Sym}^2(\sigma)$ extends to a representation of W'_F with values in $\mathrm{O}(3,\mathbb{C})$ which by equation (2) must be $\omega_{E/F}\sigma_\pi$.

To conclude, theta lift of representations of $\mathrm{SL}_2(F)$ to $\mathrm{SO}(4,2)(F)$ have parameters which are as in the Offen-Sayag theorem, and that conversely, Offen-Sayag parameters come from theta lifts from $\mathrm{SL}_2(F)$.

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