PARABOLICALLY INDUCED REPRESENTATIONS OF p-ADIC G_2 DISTINGUISHED BY SO_4, I

SARAH DIJOLS

Abstract. We consider the parabolically induced representations of the symmetric space $\mathrm{SO}_4 \backslash G_2$ over a p-adic field using the geometric lemma when the inducing parabolic subgroup is either P_β or P_α. Using an explicit description of the embedding of G_2 in GL_8, we characterize precisely the induced representations which are (SO_4, χ)-distinguished, given a certain type of involutions is chosen.

Contents

1. Introduction

The aim of this paper is to identify certain parabolically induced complex representations of the exceptional group $\mathbf{G}_2(F)$, over a p-adic field F, that admit a linear functional invariant under the special orthogonal group $\mathrm{SO}_4(F)$.

In the last two decades, motivated by the study of period integrals, many works [7, 23, 22] have described the distinguished representations of various classical groups, for instance the general linear groups and the unitary groups by their symplectic, unitary or general linear subgroups. Around the same

Date: August 2, 2023.
2010 *Mathematics Subject Classification.* Primary 22E50; Secondary 20G41.
Key words and phrases. Parabolic induction, distinguished representations, G_2.

time, the far-reaching Sakellaridis-Venkatesh conjectures have reignited interest and gave further motivations in the description and classification of representations of p-adic symmetric spaces (as a particular instance of spherical varieties) G/H.

In this realm of research, very little has been understood regarding exceptional groups, a recent work of Gan-Gomez [8], dealt with many low-rank varieties, including $\mathbf{G}_2/\mathrm{SL}_3$ (a spherical variety which is not a symmetric space). Their work, however, does not deal with a precise description or classification of representations of $\mathbf{G}_2(F)$ which might be distinguished by $\mathrm{SL}_3(F)$. Indeed, such classification would require to use the geometric lemma method (also known as "orbit method" since it relies on analysing consecutively a set of parabolic orbits). Our paper constitutes the first instance of its application (implementable if the quotient is a symmetric space, or in the Galois case) to an exceptional group. The main tools in our investigation have been exposed in [20]. The drawback of our approach is that it allows us to *only* deal with parabolically induced representations.

The strategy described in the paper of Offen [20] consists in reducing the question of distinction of the induced representations of G by H to a question of distinction at the level of a subgroup $L_x \subseteq M$ associated to a representative x for each parabolic orbit. It involves computing the relevant subgroups $Q_x = L_x \rtimes U_x$ and associated modular character. To do so, since none of the patterns of classical groups were reproducible in our context, we have used the mathematical software **SageMath** and an explicit embedding of $\mathbf{G}_2$ into GL_8. In this paper, we deal with the case of the two maximal parabolic subgroups P_β and P_α of $\mathbf{G}_2$, where β stands for the long root. Following a method of [10], we first describe the twelve or thirteen double cosets' representatives and therefore will study as many parabolic orbits. To implement the subsequent steps in *SageMath*, we need to write an explicit expression of the Levi subgroup M_β (resp. M_α) using Bruhat cells, identify the admissible orbits, and verify their closedness or openess. We have also identified the matching elements in $W_\beta \backslash W / W_\beta$ for each orbit representative and the result is given in the Appendix. Finally, the results where we identified the Levi subgroups $L = M \cap w_x M w_x^{-1}$, for each matching element w_x, and also implemented various computations to check properties of the orbits (see in [20]) using a modified version of the orbit representatives have been included in the form of codes. A better strategy was eventually found using a stricter definition of admissibility (see Definition 3.5), already offered in the literature [21]. The software **SageMath** was used to do the matching of double coset representatives with elements in $W_\beta \backslash W / W_\beta$, to check the (strict)-admissibility, the openess and closedness conditions, and deduce what would be the subgroups L, L_x, U_x.

Our main results are the following:

Theorem (Closed orbit)**.** *Let χ be a character of $\mathrm{SO}_4(F)$. It is a quadratic character of $F^\times$. It can be seen as a character of GL_2 (those are given by $\chi \circ \det$ for a quasi-character χ of $F^\times$). Let P_β (resp. P_α) denote the maximal parabolic corresponding to the root β (resp. α). The parabolic induced representations of G_2 which are (SO_4, χ)-distinguished include the following representations:*

- *The induction from P_β to G_2 of the reducible principal series $I(\chi\delta_{P_\beta}^{1/2}|.|^{-1/2} \otimes |.|)$ of GL_2.*
- *The induction from P_α to G_2 of the reducible principal series $I(\chi\delta_{P_\alpha}^{1/2}|.|^{1/2} \otimes \chi\delta_{P_\alpha}^{1/2}|.|^{-1/2})$ of GL_2.*
- *The induced representation $I_{P_\beta}^{G_2}((\chi \circ \det)\delta_{P_\beta}^{1/2})$*
- *The induced representation $I_{P_\alpha}^{G_2}((\chi \circ \det)\delta_{P_\alpha}^{1/2})$.*

Theorem (Distinguished induced parabolic representations and admissible orbits)**.** *We take the involution θ defining $\mathrm{SO}_4(F) = \mathbf{G}_2^\theta(F)$ to be of the form θ_{t_i} for $i \in \{0, 1, 2\}$, as well as the expressions of the Levi subgroups as defined in the Subsection 7.1. The parabolically induced representations from the parabolic subgroups P_β or P_α of G_2 distinguished by SO_4 whose linear forms arise from admissible, open or closed, orbits are necessarily of the form given in the Theorems 8.2 and 8.3.*

Our computations also reveal a mysterious and exciting phenomenon with the open orbits which are parametrized by the number of quadratic extensions E of F, see the Proposition 7.5.
The context of dealing with the split exceptional group $\mathbf{G}_2$ gives to this paper its computational (via *SageMath*) nature. All our codes and *SageMath* computation are available at the following link: `https://github.com/sarahdijols/G2SO4`. It is worth mentioning that this software helps us only to multiply many 8-dimensional matrices, but all these multiplications could be, in principle, done by hand. No programming skills are needed to understand the codes available at this link. A byproduct of the strategy developed in this work is to provide explicit expressions of the tori, roots subgroups, and Levi subgroups of G_2, which allow, for instance, to compute the modulus for the maximal parabolic subgroups and the Borel of G_2. We believe these codes could be useful to the math community.

Here we briefly outline the contents of the paper. In Section 2, we establish notation and recall some basic definitions. Section 3 contains a review of two key results proved by Offen in [20], and Section 4 provides some result on the distinguished representations that form the inducting data for the representations of $\mathbf{G}_2(F)$ studied here. We study the structure of the symmetric space $\mathbf{G}_2(F)/\mathrm{SO}_4(F)$ in Section 5; additional detail is provided in Appendix A, where an embedding of $\mathbf{G}_2(F)$ into $\mathrm{GL}_8(F)$ is discussed. In Section 6, we describe the double cosets and double cosets representatives, while in Section 7, we study the orbits in $\mathbf{G}_2(F)/\mathrm{SO}_4(F)$ under the twisted action of standard parabolic subgroups of $\mathbf{G}_2(F)$. Finally, the main results on $\mathrm{SO}_4(F)$-distinguished parabolically induced representations of $\mathbf{G}_2(F)$ are stated and proved in Section 8.

We, finally, mention here that this paper considers only sufficient conditions for distinction, as presented in the Propositions 7.1 and 7.2 of [20]. The necessary conditions which may involve using Proposition 4.1 in [20] will be addressed, to the greatest extent possible, in our subsequent work. The reader will notice that all the ingredients have been prepared to do so in the form of codes, as the algorithm to compute the expressions for the subgroups $L_x \subset M$ and the relevant modular characters have been written and tested (see the files "delta-functions-Pb-Pa-min.ipynb" and "delta-functions-Q-x-clean(1).ipynb" in particular).

Acknowledgements. The author would like to express her gratitude to Steven Spallone for communicating, in its entirety, the material in Appendix A on the structure of $\mathbf{G}_2$ as the group of automorphisms of the Cayley algebra. She also thanks Qing Zhang and David Ginzburg for giving her a few precious references and hints on $\mathbf{G}_2$, and Dipendra Prasad for some general comments. She would like to warmly thank Arnab Mitra and Jerrod Smith for many discussions on the method and on the earlier calculations done in this work, and Jerrod Smith for his contribution to a few files in the code, and to the organization of this paper. The author has benefited from very good work conditions at the YMSC, Beijing, and at the MPIM, Bonn.

2. Notation and Preliminaries

Let F be a non-Archimedean local field of characteristic zero and odd residual characteristic. Let $\mathcal{O}_F$ be the ring of integers of F with prime ideal $\mathfrak{p}_F$. Fix a uniformizer ϖ of F; note that $\mathfrak{p}_F = \varpi\mathcal{O}_F$. Let q be the cardinality of the residue field $k_F = \mathcal{O}_F/\mathfrak{p}_F$. Let $|\cdot|_F$ denote the normalized absolute value on F such that $|\varpi|_F = q^{-1}$. We write $|\cdot|$ for the usual absolute value on the field $\mathbb{C}$ of complex numbers.

Let $G = \mathbf{G}(F)$ be the F-points of a connected reductive group defined over F. We let e denote the identity element of G. For any $g \in G$, we denote the inner F-automorphism of G given by conjugation by g by Int_g. That is, $\mathrm{Int}_g(x) = gxg^{-1}$ for all $x \in G$. Recall that the map $\mathrm{Int} : G \to \mathrm{Aut}_F(G)$ given by $g \mapsto \mathrm{Int}_g$ is a group homomorphism. Moreover, $\ker(\mathrm{Int}) = Z_G$ is the centre of G. Note that if $g^2 = e$, then Int_g is an involution, that is, an order two automorphism. Indeed, if $g^2 = e$, then for any $x \in G$

$$(\mathrm{Int}_g)^2(x) = \mathrm{Int}_g \circ \mathrm{Int}_g(x) = g^2xg^{-2} = exe = x,$$

and $(\mathrm{Int}_g)^2 = \mathrm{Id}_G$ is the identity map on G. Observe that $x \in G$ is fixed by Int_g if and only if $x \in C_G(g)$, where $C_G(g)$ is the centralizer of g in G.

All representations are over complex vector spaces. We will often abuse notation and refer to a representation (π, V) of G simply as π. We write

$\mathbf{1}_G : G \to \mathbb{C}^\times$ for the trivial character of G, that is, $\mathbf{1}_G(g) = 1$ for all $g \in G$. We assume that all representations (π, V) of G are smooth in the sense that the stabilizer of any vector $v \in V$ is an open subgroup of G. A character of G is a one-dimensional smooth representation of G (not necessarily unitary).

Let P be a parabolic subgroup of G. Let N be the unipotent radical of P, and let M be a Levi subgroup of P. Let $\delta_P : P \to \mathbb{R}_{>0}$ be the modular character of P. Recall that $\delta_P(p) = |\det \mathrm{Ad}_{\mathfrak{n}}(p)|_F$ for all $p \in P$, where $\mathrm{Ad}_{\mathfrak{n}}$ denotes the adjoint action of P on the Lie algebra $\mathfrak{n}$ of N [5]. Given a smooth representation (σ, W) of M, we denote the normalized parabolic induction of σ along P by $I_P^G(\sigma) = \mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \sigma)$.

2.1. **Distinguished representations.** Let H be a closed subgroup of G, and let χ be a character of H. Let (π, V) be a smooth representation of G.

Definition 2.1. The representation (π, V) is said to be (H, χ)-distinguished if there exists a nonzero linear functional λ in $\mathrm{Hom}_H(\pi, \chi)$.

If (π, V) is $(H, \mathbf{1}_H)$-distinguished, then we will simply say that (π, V) is H-distinguished. The H-distinguished representations of G are precisely those representations of G that are relevant to the study of harmonic analysis on the quotient G/H. Indeed, given a nonzero H-invariant linear functional λ in $\mathrm{Hom}_H(\pi, \mathbf{1}_H)$ the linear transformation sending $v \in V$ to the function $\varphi_{\lambda,v}$, where $\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle$ for all $g \in G$, defines an intertwining operator from (π, V) to the regular representation of G on the smooth complex valued functions on G/H. Moreover, any such intertwining operator arises this way. In studying distinguished parabolically induced representations it is necessary to consider (H, χ)-distinguished representations at the level of the inducing data.

The following elementary result is quite useful.

Lemma 2.2. *Let (π, V) be a representation of G. Suppose that π admits a central character ω_π. Let χ be a character of H. If π is (H, χ)-distinguished, then $\chi|_{H \cap Z} = \omega_\pi|_{H \cap Z}$.*

Proof. Since π is (H, χ)-distinguished, there exists a nonzero linear functional λ in $\mathrm{Hom}_H(\pi, \chi)$. Let $v \in V$ so that $\langle \lambda, v \rangle \neq 0$. Suppose that $z \in H \cap Z$. Then since λ is H-invariant and π has central character ω_π we have that

$$\chi(z)\langle \lambda, v \rangle = \langle \lambda, \pi(z)v \rangle = \langle \lambda, \omega_\pi(z)v \rangle = \omega_\pi(z)\langle \lambda, v \rangle.$$

Therefore,

$$0 = (\chi(z) - \omega_\pi(z))\langle \lambda, v \rangle$$

and since $\langle \lambda, v \rangle \neq 0$ it follows that $\chi(z) = \omega_\pi(z)$. Therefore, the restriction $\chi|_{H \cap Z}$ of χ to $H \cap Z$ agrees with the restricted central character $\omega_\pi|_{H \cap Z}$. $\square$

3. Distinction for parabolically induced representations

Here we recall the general results of Offen [20] that we utilize below. We use mostly the same notations as Offen.

Let $G = \mathbf{G}(F)$ be the F-points of a connected reductive group $\mathbf{G}$ defined over F. Let θ be an F-rational involution of $\mathbf{G}$. Let $H = \mathbf{G}^{\theta}(F)$ be the F-points of the θ-fixed set $\mathbf{G}^{\theta}$ in $\mathbf{G}$. Let $X = \{g \in G : g\theta(g) = e\}$. Elements of the set X are referred to as the θ-split elements in G. The set X carries a G-action given by

$$(g, x) \mapsto g \cdot x = gx\theta(g)^{-1}$$

for all $g \in G$ and $x \in X$. Of course, $e\theta(e) = e$, so the identity element of G lies in X. The stabilizer of e under the G-action on X is the subgroup H of θ-fixed points. It follows that the map $G \to X$ given by $g \mapsto g \cdot e$ defines an embedding of the symmetric space G/H in X as the G-orbit of the identity.

Let $x \in X$ be a θ-split element of G. The F-rational automorphism θ_x of G defined by

$$\theta_x(g) = x\theta(g)x^{-1} \qquad \text{for all } g \in G$$

is an involution. For any subgroup K of G let $K_x = \mathrm{Stab}_K(x)$ be the stabilizer of x in K for the G action on X. Then $H = G_e$ and $K_x = K^{\theta_x}$ for any subgroup K of G and $x \in X$; however, K need not be θ_x-stable so it is convenient to note that $K_x = (K \cap \theta_x(K))^{\theta_x}$.

We will assume that $\mathbf{G}$ is split over F. Let B be a Borel subgroup of G with unipotent radical N. By [11, Lemma 2.4] there exists a θ-stable maximal F-split torus T of G contained in B. We have that $B = TN$. A parabolic subgroup P of G is standard if it contains the Borel subgroup B. Suppose that P is a standard parabolic subgroup of G, then P admits a unique Levi subgroup M that contains T. Let U be the unipotent radical of P. We will always work with a standard Levi factorization $P = MU$ with $T \subseteq M$. Let $N_{G,\theta}(M) = \{g \in G : M = g\theta(M)g^{-1}\}$.

Let χ be a character of H and let $\eta \in G$. Write $\chi^{\eta^{-1}}$ for the character of $\eta^{-1}H\eta$ given by $\chi^{\eta^{-1}}(h') = \chi(\eta h' \eta^{-1})$ for all $h' \in \eta^{-1}H\eta$.

The following proposition deals with the case of a closed orbit.

Proposition 3.1 (For instance Proposition 7.1 in [20])**.** *Let χ be a character of H. Let $P = MU$ be a standard parabolic subgroup of G with unipotent radical U and Levi factor M. Let (σ, W) be a smooth representation of M. Suppose that $\eta \in G$ so that $x = \eta \cdot e \in N_{G,\theta}(M)$ and $\theta_x(P) = P$. If σ is $(M_x, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$-distinguished, then $I_P^G(\sigma)$ is (H, χ)-distinguished.*

An analogous result whose proof relies on the work of Blanc and Delorme [2] is the following:

Proposition 3.2 (Proposition 7.2 in [20])**.** *Let $P = MU$ be a standard parabolic subgroup of G with unipotent radical U and Levi factor M. Let*

(σ, W) be a smooth representation of M with finite length. Suppose that $x \in (G \cdot e) \cap N_{G,\theta}(M)$ is an element of X such that $P \cap \theta_x(P) = M$. If σ is M_x-distinguished, then $I_P^G(\sigma)$ is H-distinguished.

When applying the above results in Section 8, it will be important for us to carefully choose representatives for the various P-orbits in X following [20, Section 3]. We discuss the parabolic orbits in the setting of $G = \mathbf{G}_2(F)$ and $H = \mathrm{SO}_4(F)$ in Section 7.3.

Finally, let us recall here a recent related result of Prasad in [24] which assures us of the existence of a generic unitary principal series representation of $\mathbf{G}_2(F)$ distinguished by $\mathrm{SO}_4(F)$.

Proposition 3.3 (Proposition 11 in [24])**.** *Let (G, θ) be a symmetric space over a finite or a non-Archimedean local field k which is quasi-split over k, thus there is a Borel subgroup B of G over k with $B \cap \theta(B) = T$, a maximal torus of G over k. If k is finite, assume that its cardinality is large enough (for a given G). Then there is an irreducible generic unitary principal series representation of $G(k)$ distinguished by $G^\theta(k)$.*

3.1. **The admissibility condition.** Let us recall from [20] the existence of a map from the set of parabolic orbits to the set of twisted involutions in the Weyl group, which is, in general, neither injective nor surjective:

$$\iota_M : P\backslash X \to {}_M W_{M'} \tau^{-1} \cap \mathcal{S}_0(\theta)$$

Here M' is also a standard Levi of G, and ${}_M W_{M'}$ the set of all $w \in W$ that are left W_M-reduced and right $W_{M'}$-reduced (a set in bijection with $P\backslash G/P'$.

Let us notice first that various definitions of *admissibility* have been given in the literature. In [20], admissibility is given by the following definition:

Definition 3.4. We say that $x \in X$ (or $P.x$) is M-admissible if $M = w\theta(M)w^{-1}$ where $w = \iota_M(P.x)$.

Whereas in [21, Section 3.2.6], a stricter definition is used:

Definition 3.5 (Strict admissibility)**.** $x \in X$ (or $P.x$) is M-admissible if $M = x\theta(M)x^{-1}$.

The terminology "strict" here refers to the fact that this definition uses directly an element $x \in X$ rather than its corresponding element in the Weyl group. Possibly, in the context of classical groups these two definitions completely agree, but in our context the set of orbits which are strictly admissible would be larger than the set of admissible orbits. Indeed as computed in the code "admissibility-with-w", only $w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha$ among the four-elements set $W_\beta\backslash W/W_\beta$ is likely to be admissible.

Let us also remark that this condition is far from subsidiary since a recent work of Offen and Matringe [17], in the case of p-adic Galois symmetric spaces, implies that the admissibility condition should be enough for a given orbit to contribute to the distinction of the induced representation space. Their result is expected to be extended to general symmetric spaces. It is therefore important to be able to determine which representatives are M-admissible. Notice, however, that our *ad hoc* expression for the Levi M_β possibly makes this verification a little loose.

Finally, to end this section, we add a comment which is best suited here: In [20], Lemma 3.2 gives us that the representatives η of the double coset in $P\backslash G/H$ can be chosen so that $x \in Lw$ (for the w as defined in 3.4), where L is a standard subgroup of M such that $L = M \cap \theta_x(M)$. Assume this is the case, and let us define Q to be the standard parabolic subgroup of G with standard Levi subgroup L and unipotent radical V.
Then when we will choose our involution θ to be θ_{t_0} (see Section 5 and Proposition 7.1 for an explanation of this notation), notice that the conditions $x \in N_{G,\theta}(M)$ in the Propositions 3.1 (and its equivalent for open orbit, as presented in [20]) and the equality $M_x = M$ are essentially the same. Furthermore, in this case, $L = M \cap \theta_x(M)$ and therefore $L = M$, so that $\delta_{Q_x} = \delta_{M \ltimes U_x}$. Therefore, in applying both of these propositions, when $\theta = \theta_{t_0}$, we are reduced to the problems of identifying the representations $\sigma \in \mathrm{Rep}(M)$ which are distinguished by a certain character of $\mathrm{GL}_2(F)$.

4. Inducing data

Both of the maximal (proper) parabolic subgroups of $\mathbf{G}_2(F)$ have Levi factor isomorphic to $\mathrm{GL}_2(F)$. In this section, we collect information regarding various distinguished representations of $GL_2 = \mathrm{GL}_2(F)$. For representations of $\mathrm{GL}_2(F)$-distinguished by a maximal F-split torus, Section 3.1.3 of [21] provides an excellent summary.

Proposition 4.1. *Let Z be the centre of GL_2, and $\chi = \chi_0 \circ \det$ a character of GL_2 for χ_0 a character of $F^\times$. A smooth indecomposable representation π of GL_2 is (GL_2, χ)-distinguished if and only if it is one of the following GL_2-representations:*

- *π is isomorphic to χ (then π is irreducible).*
- *π is a reducible principal series of the form $I(\chi_0|.|^{1/2} \otimes \chi_0|.|^{-1/2})$.*

Proof. Recall any character of GL_2 factors through det. A smooth representation π of GL_2 is (GL_2, χ)-distinguished if and only if χ occurs as a quotient of π by a GL_2-subrepresentation.
Here, we only justify the second element in the list, the other being obvious. Let us denote $Q(\chi_1, \chi_2)$ the one-dimensional quotient of the reducible principal series $I(\chi_1 \otimes \chi_2)$, then $Q(\chi_1, \chi_2) \cong \mathrm{Span}\{\chi\}$ for $\chi = \chi_0 \circ \det$ a character of $\mathrm{GL}_2(F)$. Notice that $\mathrm{Span}\{\chi\}$ is GL_2-invariant subspace of $I(\chi_1 \otimes \chi_2)$ where $\mathrm{GL}_2(F)$ acts via χ itself.

Given a character $\chi_0 \circ \det : \mathrm{GL}_2(F) \to \mathbb{C}^\times$, where χ_0 is a character of $F^\times$, by a well-known description of $\mathrm{GL}_2(F)$-representations and reducibility point of principal series (see [3], Chapter 4 or [19] Proposition 1.1) it occurs as an irreducible quotient of a representation of $\mathrm{GL}_2(F)$, if and only if the representation is the reducible principal series $I(\chi_1 \otimes \chi_2) = \mathrm{Ind}_B^G(\chi_0|.|^{1/2} \otimes \chi_0|.|^{-1/2}) = \mathrm{Ind}_B^G(\delta_B^{1/2}\chi_0 \otimes \chi_0)$. □

Remark 4.2. In G_2, we take β as the long root. Since we will consider later the parabolic $P_\beta \subset G_2$, its Levi $M_\beta \cong GL_2$ under the map $t \to (\alpha+\beta(t), \alpha(t)) = (s, ts^{-1})$ for $t, s \in F^\times$. Then the induced principal series takes the form $I(\chi_0|.|^{-1/2} \otimes |.|)$, by Proposition 1.1 in [19]. This observation can also be verified applying Lemma 2.2.

5. The exceptional group G_2 and its symmetric subgroup SO_4

Throughout the rest of this paper unless specified otherwise let $G = \mathbf{G}_2(F)$ be the group of F-points of the split exceptional group $\mathbf{G}_2$, and let $H = \mathrm{SO}_4(F)$ be the F-points the split special orthogonal group SO_4. We start with a lemma which offers an interesting geometrical interpretation of the subgroup H, under certain conditions.

Lemma 5.1. *Let us assume the characteristic of the field F is different from 2. Let $\mathcal{C}$ be a composition algebra of dimension 8, D a quaternion subalgebra, $a \in D^\perp$, with $N(a) \neq 0$. Assume $N(a) = 1$, then the quotient $\mathbf{G}_2/\mathrm{SO}_4$ is the space of quaternionic subalgebras of $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a composition algebra (in our context, of dimension 8 over F, for instance the octonions, $\mathbb{O}$), and D be a finite dimensional composition subalgebra of $\mathcal{C}$. Suppose $a \in D^\perp$, with $N(a) \neq 0$ then $D_1 = D \oplus Da$ and D_1 is a composition subalgebra. The subalgebra D_1 is said to be constructed by doubling from D. The norm is given by $N(x+ya) = N(x) - \lambda N(y)$, for x, y in D, and $\lambda = -N(a)$. For instance the split octonion (see the Appendix) can be constructed from the split quaternion algebra by such doubling process as in Proposition 1.5.1, [27].
Let now assume this composition $\mathcal{C}$ is an octonion algebra, and D a given quaternion subalgebra. If one chooses a to be of norm one, then SO_4 is seen as the group $G_D = \{\sigma \in G = \mathrm{Aut}(\mathcal{C}) : \sigma(D) = D\}$ and the argumentation goes as follows: Since G_D preserves D, it also preserves the orthogonal complement aD. If $\sigma \in G_D$ acts trivially on aD, then G_D fixes a so $\sigma(ua) = \sigma(u)a = ua$ and so σ acts trivially on D as well, so $\sigma = 1$. Thus G_D acts faithfully on aD (but not on D) and we have an injective homomorphism $G_D \hookrightarrow O(4)$.

It remains to show that G_D is of dimension six. To do so one observes that the restriction map from $G_D \to \mathrm{Aut}(D) \cong \mathrm{SO}(D_0)(F)$ (here D_0 are the trace zero elements in D) is surjective by an application of Corollary 1.7.3 in [27], and let K be the kernel of this map. Proposition 2.2.1 in [27] tells us that $\mathbf{K}$ (the algebraic group of $\bar{F}$-automorphisms of $\mathcal{C}_{\bar{F}}$ that fix $D_{\bar{F}}$ elementwise) is a 3-dimensional algebraic group and connected. The isomorphism between $D_{\bar{F}}$ and the unitary quaternions inducing those properties induces the same isomorphism at the level of F. Let us remark that the isomorphism $\mathrm{Aut}(D) \cong \mathrm{SO}(D_0)(F)$ is due to [28], Theorem I.3.3, and using the fact (see [29] Corollaries 7.1.2 and 7.1.4 for instance) that every F-algebra automorphism of D is inner, i.e $\mathrm{Aut}_F(D) \cong D^\times/F^\times$. Thus G_D fits inside the exact sequence:

$$1 \to K \to G_D \to \mathrm{SO}(D_0) \to 1$$

In particular G_D is connected and $\dim G_D = 6$, so $G_D \cong \mathrm{SO}_4$. This result is true if $D = \mathbb{H}$ and $D_1 = \mathbb{O}$, and holds in a p-adic context with the additional conditions given in the statement of this lemma. □

Remark 5.2. Notice that in our context, and to embed $\mathbf{G}_2$ into GL_8 (see the Appendix), we have chosen $N(a) = -\lambda = -1$. It would be interesting to consider the embedding of $\mathbf{G}_2$ into GL_8 using $N(a) = 1$ and proceed with the remaining steps using this convention.

Let T be a maximal F-split torus of G. Let B be a Borel subgroup of G containing T and let N be the unipotent radical of B. Then $B = TN$ is a Levi decomposition of B. A parabolic subgroup P of G is standard if it contains the fixed Borel subgroup B. The standard Levi factor M of a standard parabolic P is the unique Levi factor that contains the torus T. Let W be the Weyl group of G defined with respect to T.

Recall that $\mathbf{G}_2$ is simply connected (see [18, Ch. 24] for instance). With this fact, one can adjust the results used in the proof of [14, Lemma 3.2(i)] to see that all elements of order 2 in G are conjugate in G. Moreover, the centralizer of an order-two element in G is isomorphic to H. The two key modifications are to use (1) the fact that the centralizer of a (finite order) semisimple element in a connected group is connected (this is a theorem of Springer and Steinberg, see [13, Theorem 2.11]), and (2) all maximal F-split F-tori in a smooth connected group are conjugate over the F-points of the group (this is a theorem of Borel and Tits, see [6, Theorem C.2.3]).

Let $\theta = \mathrm{Int}(t_0)$, where $t_0 \in T$ is an order two element (for instance, we can take $t_0 = \gamma(1, -1)$, using the notation of Appendix A). Since, $t_0^2 = e$, the inner automorphism θ is an involution. Observe that since $t_0 \in T$, the torus T and Borel subgroup B are θ-stable. The group G^θ of F-points of the θ-fixed points in G is the centralizer of t_0 in G, and so $G^\theta \cong H$.

Remark 5.3. Note that T is θ-stable. It follows that θ induces an involution on the Weyl group W which we also denote by θ.

As above, let $X = \{g \in G : g\theta(g) = e\}$. Recall that the set X carries a G-action given by

$$(g, x) \mapsto g \cdot x = gx\theta(g)^{-1}$$

for all $g \in G$ and $x \in X$. Of course, $e\theta(e) = e$, so the identity element of G lies in X. The stabilizer of $e \in X$ under the G-action is the subgroup G^θ of

θ-fixed points. The map $G \to X$ given by $g \mapsto g \cdot e$ defines an embedding of the symmetric space G/H in X as the G-orbit of the identity.

Lemma 5.4. *The set X is a disjoint union of two G-orbits, namely $G \cdot e$ and the singleton set $\{t_0\}$.*

Proof. By definition,

$$X = \{g \in G : g\theta(g) = e\} = \{g \in G : gt_0g = t_0\}.$$

The G-orbit of the identity element is

$$G \cdot e = \{g \cdot e : g \in G\} = \{gt_0g^{-1}{t_0}^{-1} : g \in G\}.$$

In particular, for all $g \in G$, $g \cdot e = gt_0g^{-1}{t_0}^{-1} \in X$. On the other hand, $t_0 \in X$ but t_0 is not in $G \cdot e$. Indeed, since ${t_0}^2 = e$ we have $t_0t_0t_0 = t_0e = t_0$ so $t_0 \in X$. Now argue by contradiction and suppose that $t_0 = g \cdot e$ for some $g \in G$. It follows that

$$e = {t_0}^2 = (g \cdot e)t_0 = gt_0g^{-1}{t_0}^{-1}t_0 = gt_0g^{-1},$$

and $t_0 = g^{-1}eg = e$ which contradicts that $t_0 \neq e$ is an order two element of T. Thus, $G \cdot e \cap \{t_0\} = \varnothing$. Moreover, $\{t_0\}$ is a G-orbit in X because t_0 is fixed under the G action on X. Indeed, for any $g \in G$

$$g \cdot t_0 = gt_0\theta(g)^{-1} = gt_0t_0g^{-1}{t_0}^{-1} = geg^{-1}{t_0}^{-1} = {t_0}^{-1} = t_0.$$

Finally, we show that X is the union of $G \cdot e$ and $\{t_0\}$. Suppose that $x \in X$. Then $xt_0x = t_0$. Thus

$$(xt_0)^2 = xt_0xt_0 = {t_0}^2 = e.$$

Therefore, xt_0 is either the identity or an order two element of G. If $xt_0 = e$, then $x = t_0^{-1} = t_0 \in \{t_0\}$. Otherwise, xt_0 has order two and by [14, Lemma 3.2(i)] (and the remarks above) xt_0 is G-conjugate to t_0. In the latter case, there exists $g \in G$ so that $g^{-1}xt_0g = t_0$, that is, $x = gt_0g^{-1}{t_0}^{-1} = g \cdot e$. Therefore, $x \in \{t_0\}$ or $x \in G \cdot e$ and $X = G \cdot e \cup \{t_0\}$ is a union of (disjoint) G-orbits. □

5.1. **Roots and Weyl groups.** Let $\Delta = \{\alpha, \beta\}$ be a basis of the root system Φ of G with respect to T where α is the short root and β is the long root. The set of positive roots of $\mathbf{G}_2$ is

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

Let us recall that that we denote $W = N_G(T)/T$ the Weyl group of $\mathbf{G}_2$. More generally, for a standard Levi subgroup M of $\mathbf{G}_2$, we denote $W_M = N_M(T)/T$ the Weyl group of M with respect to T.

The Weyl group of $\mathbf{G}_2$ is generated by the simple reflections w_α and w_β attached to the roots α and β. In particular, W is a finite group of size 12 and we can realize W as follows:

$$W = \{e, w_\alpha, w_\beta, w_\alpha w_\beta, w_\beta w_\alpha, w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha, w_\beta w_\alpha w_\beta w_\alpha, w_\alpha w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha w_\beta w_\alpha, w_\beta w_\alpha w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta\}.$$

We summarize the action of the simple reflections w_α and w_β on Φ^+ in Figure 5.1.

FIGURE 5.1. Action of w_α and w_β on Φ^+

Φ^+	α	β	$\alpha+\beta$	$2\alpha+\beta$	$3\alpha+\beta$	$3\alpha+2\beta$
$w_\alpha \cdot \Phi^+$	$-\alpha$	$3\alpha+\beta$	$2\alpha+\beta$	$\alpha+\beta$	β	$3\alpha+2\beta$
$w_\beta \cdot \Phi^+$	$\alpha+\beta$	$-\beta$	α	$2\alpha+\beta$	$3\alpha+2\beta$	$3\alpha+\beta$

For each root $\gamma \in \Phi$, let U_γ be the associated root subgroup in $\mathbf{G}_2$ and fix an isomorphism $x_\gamma : F \to U_\gamma$. For $g_1, g_2 \in \mathbf{G}_2$, let $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$. For all $x, y \in F$, we have the following commutator relations (see, for instance, [25, pp. 443]),

$$\begin{aligned}
[x_\alpha(x), x_\beta(y)] &= x_{\alpha+\beta}(-xy) x_{2\alpha+\beta}(-x^2y) x_{3\alpha+\beta}(x^3y) x_{3\alpha+2\beta}(-2x^3y^2) \\
[x_\alpha(x), x_{\alpha+\beta}(y)] &= x_{2\alpha+\beta}(-2xy) x_{3\alpha+\beta}(3x^2y) x_{3\alpha+2\beta}(3xy^2) \\
[x_\alpha(x), x_{2\alpha+\beta}(y)] &= x_{3\alpha+\beta}(3xy) \\
[x_\beta(x), x_{3\alpha+\beta}(y)] &= x_{3\alpha+2\beta}(xy) \\
[x_{\alpha+\beta}(x), x_{2\alpha+\beta}(y)] &= x_{3\alpha+2\beta}(3xy).
\end{aligned}$$

For all remaining pairs of positive roots γ_1, γ_2, we have $[x_{\gamma_1}(x), x_{\gamma_2}(y)] = e$.

We may realize the group $H \cong \mathrm{SO}_4(F)$ as the subgroup generated by T and the images of $x_\beta, x_{2\alpha+\beta}$ (since SO_4 is chosen to be generated by β and $2\alpha+\beta$ -see, for instance, [1] in [9, pp. 137]), its Weyl group must be generated by w_β and $w_{2\alpha+\beta}$. Then the Weyl group of H with respect to T is

$$W_{\mathrm{SO}_4} = \{1, w_\beta, w_{2\alpha+\beta}, w_\beta w_{2\alpha+\beta}\}.$$

Let B_{SO_4} be the standard Borel of H with respect to the positive roots β and $2\alpha+\beta$, then the set B/B_{SO_4} has representatives

$$\{x_\alpha, x_{\alpha+\beta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\}.$$

For $r_i \in F, i = 1, 2, 3, 4$, write:

$$[r_1, r_2, r_3, r_4] = x_\alpha(r_1) x_{\alpha+\beta}(r_2) x_{3\alpha+\beta}(r_3) x_{3\alpha+2\beta}(r_4)$$

6. Computation of the double cosets representatives

The set $B\backslash X$ of B-orbits in X is finite [11, Proposition 6.15]; therefore, $B\backslash G/H$ is finite [11, Corollary 6.16]. In particular, for any standard parabolic subgroup of a (p-adic) reductive group G, the set $P\backslash G/H$ is a finite set.

Let $P_\alpha = M_\alpha N_\alpha$ (respectively $P_\beta = M_\beta N_\beta$) be the standard parabolic subgroup of G with Levi factor M_α and unipotent radical N_α such that $\mathrm{Im}(x_\alpha) \subseteq M_\alpha$ (respectively $\mathrm{Im}(x_\beta) \subseteq M_\beta$). Then N_α is generated by the images of $\{x_\beta, x_{\alpha+\beta}, x_{2\alpha+\beta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\}$ (respectively N_β is generated by the images of $\{x_\alpha, x_{\alpha+\beta}, x_{2\alpha+\beta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\}$). We follow a method implemented by Ginzburg in [10] to compute the double cosets representatives for the two maximal parabolic subgroups P_β and P_α.

Lemma 6.1. *Let w_0 denote the element $w_\alpha w_\beta w_\alpha w_\beta w_\alpha$, and let $r_3 \in F^\times/F^{\times 2}$. The set of representatives of $P_\beta\backslash G_2/\mathrm{SO}_4$ is:*

$$\{e, w_\alpha, w_\alpha x_\alpha(1), w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1), w_\alpha w_\beta w_\alpha x_\alpha(1), w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1),$$
$$w_0 x_{\alpha+\beta}(1), w_0 x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)\}$$

Proof. The set of representatives for $P_\beta\backslash G_2/B$ is

$$A = \{e, w_\alpha, w_\alpha w_\beta, w_\alpha w_\beta w_\alpha, w_\alpha w_\beta w_\alpha w_\beta, w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha\}$$

Notice that we have used that the last element in W_{G_2} has order two hence is equal to the other order two element whose action is the same on all roots : $w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta = w_\beta w_\alpha w_\beta w_\alpha w_\beta w_\alpha$. The set B/B_{SO_4} is

$$\{x_\alpha(r_1), x_{\alpha+\beta}(r_2), x_{3\alpha+\beta}(r_3), x_{3\alpha+2\beta}(r_4)\}$$

A complete set of representatives of $P_\beta\backslash G_2/B_{\mathrm{SO}_4}$ is given by:

$$\mathcal{S} := \{w[r_1, r_2, r_3, r_4], w \in A, r_i \in F\}$$

In the subsequent step, we will use two tricks to find equivalences between different elements of $\mathcal{S}$:

- We will rescale the unipotent element from r_i to 1 using a torus element. If $r_1 \neq 0$, we can find a torus element t such that $x_\alpha(r_1) = t x_\alpha(1) t^{-1}$; since $w_\alpha t w_\alpha^{-1}$ in P_β and t in SO_4 , we get $w_\alpha x_\alpha(r_1) \sim w_\alpha x_\alpha(1)$. Notice that there also exists a torus element which rescales a product of two root subgroups.
- We use the commutator relations given in the previous subsection, along with the expressions given in the Table 5.1 to simplify the expressions for each $w \in A$.

Write $x \sim y$ if x and y are in the same double coset in $P_\beta\backslash G_2/\mathrm{SO}_4$.

Since $x_\alpha(r_1)x_{\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)$ belong to N_{P_β}, we have $e.[r_1, r_2, r_3, r_4] \sim e$, i.e they are in the same double coset in $P_\beta\backslash G_2/\mathrm{SO}_4$. For instance, consider $w_\alpha x_{3\alpha+2\beta}(r_4)x_{3\alpha+\beta}(r_3)x_{\alpha+\beta}(r_2)x_\alpha(r_1)$, since $w_\alpha x_{3\alpha+2\beta} x_{\alpha+\beta} w_\alpha^{-1}$ in N_β, $w_\alpha x_{3\alpha+\beta} \in M_\beta$, what remains is $w_\alpha x_\alpha$. The same logic applies to reduce $w_\alpha w_\beta x_{3\alpha+2\beta}(r_4)x_{3\alpha+\beta}(r_3)x_{\alpha+\beta}(r_2)x_\alpha(r_1)$ to $w_\alpha w_\beta x_{\alpha+\beta} x_\alpha$. Since $x_{3\alpha+\beta}(1)$ and $x_{\alpha+\beta}(1)$ commute, we obtain $w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1)$ and we also have $w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1)x_{\alpha+\beta}(1)$. The last representative $w_0[r_1, r_2, r_3, r_4]$ will be dealt with in the last part of this proof.

(6.1)

$$\{e, w_\alpha, w_\alpha x_\alpha(1); w_\alpha w_\beta, w_\alpha w_\beta x_{\alpha+\beta}(1)x_\alpha(1), w_\alpha w_\beta x_{\alpha+\beta}(1), w_\alpha w_\beta x_\alpha(1);$$
$$w_\alpha w_\beta w_\alpha, w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1); w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1); w_\alpha w_\beta w_\alpha x_\alpha(1);$$
$$w_\alpha w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1)x_{\alpha+\beta}(1), w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1), w_\alpha w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1),$$
$$w_0 x_{\alpha+\beta}(1), w_0 x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1), w_0[0,1,r_3,0]\}$$

The second step in this procedure is to look at these elements, as compared to the set W_{SO_4} and try to simplify further:

$$w_\alpha w_\beta \sim w_\alpha$$

$$w_\alpha w_\beta x_{\alpha+\beta}(1)x_\alpha(1) \sim w_\alpha x_\alpha(1)w_\beta x_\alpha(1) \sim w_\alpha x_\alpha(1)x_{\alpha+\beta}(1)w_\beta \sim w_\alpha x_\alpha(1)x_{\alpha+\beta}(1)$$
$$w_\alpha w_\beta x_{\alpha+\beta}(1) \sim w_\alpha x_\alpha(1)w_\beta \sim w_\alpha x_\alpha(1); w_\alpha w_\beta x_\alpha(1) \sim w_\alpha x_{\alpha+\beta}(1)w_\beta \sim w_\alpha x_{\alpha+\beta}(1)$$
$$w_\alpha w_\beta w_\alpha w_\beta \sim w_\alpha w_\beta w_\alpha$$

(6.2) $w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1)x_{\alpha+\beta}(1) \sim w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1);$
$w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1) \sim w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1); w_\alpha w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1) \sim w_\alpha w_\beta w_\alpha x_a(1)$

$$w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta \sim w_\alpha w_\beta w_\alpha w_\beta w_\alpha = w_0 \in W_{\mathrm{SO}_4}$$

$w_\alpha x_\alpha(1)x_{\alpha+\beta}(1) \sim w_\alpha x_{\alpha+\beta}(1)x_\alpha(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(1)x_{3\alpha+2\beta}(1)$
$\sim$ $w_\alpha x_\alpha(1)x_{3\alpha+\beta}(1)x_{3\alpha+2\beta}(1)x_{2\alpha+\beta}(1)$ since $x_{2\alpha+\beta}(1)$ is in SO_4 it disappears. We are left with $w_\alpha x_{3\alpha+\beta}(1)x_{3\alpha+2\beta}(1)x_\alpha(1)$, and therefore $\cong w_\alpha x_\alpha(1)$.
$w_\alpha w_\beta w_\alpha \sim w_\beta w_\alpha^2 w_\beta w_\alpha w_\beta w_\alpha \sim w_\beta w_\alpha w_0 \sim w_\beta w_\alpha$ since w_0 is in W_{SO_4}.

Consider, finally, the representative $w_0[r_1, r_2, r_3, r_4]$. This one cannot be simplified using the tricks described above. However, one notices the SO_4 contains a copy of GL_2 (constituted of the $x_{\pm\beta}$ and the torus) which commutes with w_0. Looking at this representative in the quotient by SO_4 gives an action of GL_2 on $x_\alpha(r_1)x_{\alpha+\beta}(r_2)$ which is the standard action of GL_2 on a two-dimensional vector space. Under this action, there are two orbits, one with $r_1 = r_2 = 0$ and the second where $(r_1, r_2) \neq (0, 0)$. The first orbit yields the representative $w_0[0, 0, r_3, r_4]$ which, by an action of the same GL_2 on the two-dimensional vector space generated by $x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)$ yields two representatives w_0, and $w_0[0, 0, 0, 1]$.

For the second orbit, $(r_1, r_2) \neq (0, 0)$, we may assume without loss of generality, that $(r_1, r_2) = (0, 1)$, then we are reduced to $w_0[0, 1, r_3, r_4]$. Now, either $r_3 = r_4 = 0$, which yields the representative $w_0[0, 1, 0, 0]$; or $r_3 = 0$ and $r_4 \neq 0$, in which case, you can choose a torus element in SO_4 which acts linearly on $x_{2\alpha+3\beta}(r_4)$ and commutes with $x_{\alpha+\beta}(r_2)$ so we can reduce further the expression to $w_0[0, 1, 0, 1]$.
Finally, if $r_3 \neq 0$, one first conjugates by a suitable element of the form $x_\beta(m)$ the expression $w_0[0, 1, r_3, r_4]$ to obtain $w_0[0, 1, r_3, 0]$ (this is easily checked

in *SageMath*, one should obtain $m = -r_3/r_4$) , and further there exists an element of the torus t_1 such that $x_{3\alpha+\beta}(r_3)x_{\alpha+\beta}(1) = t_1 x_{3\alpha+\beta}(1)x_{\alpha+\beta}(1)t_1^{-1}$ (more specifically this torus satisfies $s = 1$, $t^2 = r_3$).

Then, observe that the torus which commutes with $x_{\alpha+\beta}(1)$ (i.e, you can check that too, it requires $s = t$) acts by a square on $x_{3\alpha+\beta}(r_3)$. Therefore this representative becomes $w_0[0, 1, r_3, 0]$ where $r_3 \in F^\times/F^{\times 2}$. To show that there a finite number of such representatives, one just needs to recall that when F is a local field, $F^\times/F^{\times 2}$ is finite. More specifically, let us denote π a prime in F a local field, $U = \mathcal{O}_F^\times$ and $U_1 = \{1 + x\pi^n | x \in \mathcal{O}_F\}$, and let us take u an element of U with the property that its image in U/U_1 is not a square. If $2 \nmid q$ then $\{1, u, \pi, \pi u\}$ form a complete set of cosets representatives for $F^\times/F^{\times 2}$. $\square$

Remark 6.2. The reader may have noticed that this set is pretty large (ten representatives!) whereas we would expect its dimension to be really smaller. The reference [10] also uses further simplifications by allowing root subgroups of negative roots (other than $x_{-\alpha}$ or $x_{-\beta}$) to appear in the simplifications. The reason why we have not allowed those root subgroups of negative roots to appear is due to our embedding in GL_8 and the fact that we would therefore need explicit embeddings of those root subgroups in GL_8 to proceed with further computations in *SageMath*. But the results of the Appendix do not tell us how to express them in GL_8.

Lemma 6.3. *Let w_0 and w_1 denote the elements $w_\alpha w_\beta w_\alpha w_\beta w_\alpha$ and $w_\beta w_\alpha w_\beta w_\alpha$ respectively, and let $r_3 \in F^\times/F^{\times 2}$. The set of representatives of $P_\alpha\backslash G_2/\mathrm{SO}_4$ is:*

$$\begin{aligned}\{e, w_\beta w_\alpha, w_\beta w_\beta x_\alpha(1), w_\beta w_\alpha x_{3\alpha+\beta}(1),\\ w_\beta w_\alpha x_\alpha(1)x_{3\alpha+\beta}(1), w_\beta w_\alpha w_\beta w_\alpha x_\alpha(1), w_\beta w_\alpha w_\beta w_\alpha x_\alpha(1)x_{3\alpha+2\beta}(1),\\ w_\beta w_\alpha w_\beta w_\alpha x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}x_{3\alpha+\beta}(r_3)\}\end{aligned}$$

Proof. First, notice that the set $P_\alpha\backslash G_2/B \cong W_\alpha\backslash W$ can be described by $\{e, w_\beta, w_\beta w_\alpha, w_\beta w_\alpha w_\beta, w_\beta w_\alpha w_\beta w_\alpha, w_\beta w_\alpha w_\beta w_\alpha w_\beta\}$. We apply the same strategy than in Lemma 6.1. Then, we start with the identity element in $W_\alpha\backslash W$ and simplify the expression of the representative in $P_\alpha\backslash G_2/\mathrm{SO}_4$:

$$e x_\alpha(r_1)x_{\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4) \sim e$$

since $x_\alpha(r_1)$ belongs to M_α and $x_{\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)$ belongs to N_{P_α}.

Secondly, we take the next element in $W_\alpha\backslash W$, and let it acts on each of the root subgroup of $[r_1, r_2, r_3, r_4]$, the result lies in P_α. So $w_\beta[r_1, r_2, r_3, r_4] \sim w_\beta$.

Other simplifications include: $w_\beta w_\alpha[r_1, r_2, r_3, r_4] \sim w_\beta w_\alpha[r_1, r_3, r_4, r_2] \sim w_\beta w_\alpha[r_3, r_4, r_1, r_2] \sim w_\beta w_\alpha x_\alpha(r_1)x_{3\alpha+\beta}(r_3) \sim w_\beta w_\alpha x_\alpha(1)x_{3\alpha+\beta}(1)$ since $w_\beta w_\alpha x_{3\alpha+2\beta}(r_4)$ lands in P_α, and $w_\beta w_\alpha x_{\alpha+\beta}(r_2) = x_{2\alpha+\beta}(r_2) \in W_{\mathrm{SO}_4}$.

Then comes $w_\beta w_\alpha w_\beta[r_1, r_2, r_3, r_4]$ which is equivalent to $w_\beta w_\alpha w_\beta x_{\alpha+\beta}(r_2)x_{3\alpha+2\beta}(r_4)$.

The expression $w_\beta w_\alpha w_\beta w_\alpha[r_1,r_2,r_3,r_4]$ does not simplify, but we can multiply on the left by w_α and get $w_\alpha w_\beta w_\alpha w_\beta w_\alpha[r_1,r_2,r_4,r_3]$, which appears already in Lemma 6.1. Finally, the element $w_\beta w_\alpha w_\beta w_\alpha w_\beta[r_1,r_2,r_3,r_4] \sim w_\beta w_\alpha w_\beta w_\alpha[r_2,r_1,r_4,r_3]$, letting w_β acts on each root subgroup and using $w_\beta \in W_{\mathrm{SO}_4}$. Further, since $w_\alpha \in P_\alpha$, we can again multiply on the left by w_α and get again the expression already studied in Lemma 6.1: $w_\alpha w_\beta w_\alpha w_\beta w_\alpha[r_2,r_1,r_4,r_3]$. Using the same argumentation (to deal with this more challenging representative) as in the proof of this Lemma, this element yields the representatives:

$$w_\beta w_\alpha w_\beta w_\alpha x_{3\alpha+2\beta}(r_4) \sim w_\beta w_\alpha w_\beta w_\alpha x_{3\alpha+2\beta}(1)$$
$$w_\beta w_\alpha w_\beta w_\alpha x_\alpha(r_2) \sim w_\beta w_\alpha w_\beta w_\alpha x_\alpha(1)$$
$$w_\beta w_\alpha w_\beta w_\alpha x_\alpha(r_2)x_{3\alpha+2\beta}(r_4) \sim w_\beta w_\alpha w_\beta w_\alpha x_\alpha(1)x_{3\alpha+2\beta}(1)$$
$$w_0 x_{\alpha+\beta}x_{3\alpha+\beta}(r_3) \quad \text{with} \quad r_3 \in F^\times/F^{\times^2}$$

As a last step, we will simplify by what could lands in W_{SO_4} on the right. For instance:

$$w_\beta \in W_{\mathrm{SO}_4}, w_\beta \sim e$$
$$w_\beta w_\alpha w_\beta \sim w_\beta w_\alpha$$
$$w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1) \sim w_\beta w_\alpha x_\alpha(1) w_\beta \sim w_\beta w_\alpha x_\alpha(1)$$
$$w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1) \sim w_\beta w_\alpha x_{3\alpha+\beta}(1) w_\beta \sim w_\beta w_\alpha x_{3\alpha+\beta}(1)$$
$$w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1) \sim w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1)w_\beta^{-1}w_\beta x_{3\alpha+2\beta}(1)w_\beta^{-1}w_\beta \sim w_\beta w_\alpha x_\alpha(1)x_{3\alpha+\beta}(1)$$

$$w_\beta w_\alpha w_\beta w_\alpha w_\beta \sim w_\beta w_\alpha w_\beta w_\alpha$$

Since we can multiply by $w_\alpha \in P_\alpha$ on the left and $w_0 \in W_{\mathrm{SO}_4}$, we have $w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha \sim e$ so $w_\beta w_\alpha w_\beta w_\alpha \sim e$.

□

7. Analysis of the orbits

7.1. **Conventions for the torus and the Levi subgroups.** For the rest of this paper let us set the following notation $\mathrm{GL}_2 = \mathrm{GL}_2(F)$.

Let t and s be F-variables. There exist two conventions to write the torus in M_β in the literature (see for instance [19] and [16]). we are writing the torus in M_β, as $T_{\mathrm{GL}_2} = \begin{pmatrix} s & 0 \\ 0 & ts^{-1} \end{pmatrix}$, so that $\beta(T_{\mathrm{GL}_2}) = e_1 - e_2 = s^2t^{-1}$. Therefore, from the Appendix A which defines the embedding of $\mathbf{G}_2$ into GL_8 (see in particular the Equation A.1), the embedding of the torus $\begin{pmatrix} s & 0 \\ 0 & ts^{-1} \end{pmatrix}$ (resp. $\begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}$) of $\mathbf{G}_2$ in GL_8, is the following:

$$T = T_{\mathrm{GL}_8} = \mathrm{diag}(1, \frac{s^2}{t}, \frac{t}{s^2}, 1, \frac{t}{s}, s, \frac{1}{s}, \frac{s}{t})$$

resp.

$$T = T_{\mathrm{GL}_8} = \mathrm{diag}(1, \frac{t}{s}, \frac{s}{t}, 1, s, t, t^{-1}, s^{-1})$$

Since most of our computations are implemented in *SageMath*, we need an explicit computable expression of the Levi $M = M_\beta$ (resp. $M = M_\alpha$). We use the Bruhat decomposition to consider these Levi subgroups as the disjoint union of the two Bruhat cells: $B.w_\beta.\overline{U_\beta}$ and $B.e.\overline{U_\beta}$ (resp. similar expressions replacing β by α), written in *SageMath* as: $U_\beta T U_{-\beta} w_\beta$ and $TU_\beta U_{-\beta}$. Let b be the F-variable entering in the matrix expression of U_β and x be the one used in $U_{-\beta}$, then the two cells are:

(7.1)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{t}{s} & 0 & 0 & \frac{tx}{s} & 0 & 0 & 0 \\ 0 & 0 & \frac{bsx}{t} + \frac{s}{t} & 0 & 0 & 0 & 0 & -\frac{bs}{t} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & bs & 0 & 0 & bsx + s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & -\frac{x}{s} & 0 & 0 & 0 & 0 & \frac{1}{s} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{tx}{s} & 0 & 0 & -\frac{t}{s} & 0 & 0 & 0 \\ 0 & 0 & -\frac{b}{s} & 0 & 0 & 0 & 0 & -\frac{bx}{s} - \frac{s}{t} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{btx}{s} + s & 0 & 0 & -\frac{bt}{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & \frac{x}{s} \end{pmatrix}$$

Let $a, b, c, d, T, u, v, w, X$ be F-variables. The following matrix would make an instance of M_β since it can be either of the two cells:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 & x \end{pmatrix} \tag{7.2}$$

In the context of M_α, we cannot deduce from the two cells' expressions $U_\alpha T U_{-\alpha} w_\alpha$ and $TU_\alpha U_{-\alpha}$ a generic expression of the Levi subgroup as given in Equation 7.2, so we use each of them separately to check the admissibility, openness and closeness conditions.

7.2. **Involutions.** In Section 5, we have shown that our involution was defined to be the conjugation by an order two element which was chosen to be a torus element of order two. Let us define three such elements by fixing the variables s and t to be ± 1. We will then let θ_{t_i} to denote the corresponding involution $\mathrm{Int}(t_i)$ on G_2 whose fixed points are SO_4:

$$t_0 = T(t = 1, s = -1)$$
$$t_1 = T(t = -1, s = 1)$$

and

$$t_2 = T(t = -1, s = -1)$$

As our reader may also be taking [20] as a reference, and since we are dealing with the definition of involution, we show that τ (used in this reference, p213) is trivial. Recall that the set of minimal semi-standard parabolic P_0 subgroups of a reductive group G forms a W-torsor. In particular, since T is θ-stable, there exists a unique Weyl element $\tau \in W$ such that $\theta(P_0) = \tau P_0 \tau^{-1}$. Applying θ to this identity yields also the condition $\theta(\tau)\tau = e$.

Proposition 7.1. *Let τ be the unique Weyl element $\tau \in W$ such that $\theta(P_0) = \tau P_0 \tau^{-1}$. then $\tau = e$ for any θ_{t_i} where θ_{t_i} is the involution defined as the conjugation by the order two element t_i.*

Proof. This is clear once one notices that the Borel subgroup $B = P_0$ is θ-stable. See the results of the computation in *SageMath*, file "tau-p213". □

7.3. **Admissible orbits, closed and open orbits.** Recall the definition of $x = \eta t_j \eta^{-1} {t_j}^{-1} \in X$ for each double coset representative η, given in Section 5. To apply the Propositions 3.1 and 3.2, we need the condition $x \in N_{G,\theta}(M)$ to hold. Notice that when we choose θ_{t_0}, this condition is just $M_x = M$. Further, the conditions of openness or closedness of parabolic orbits is verifiable by looking at $\theta_x(P)$ and P: either there are equal (closed orbit) either their intersection is the Levi M. In the code we verify these conditions with the two Bruhat cells given in 7.1, but also with the expression 7.2. In this subsection, we write all the exact properties of the orbits that could be verified using *SageMath* and the involution given by θ_{t_j}, i.e by conjugation by one of the t_j for $j \in \{0, 1, 2\}$.

Proposition 7.2. *Let us denote w_0 the element $w_\alpha w_\beta w_\alpha w_\beta w_\alpha$. Let $M = M_\beta$, and the involution be given θ_{t_j}, i.e by conjugation by one of the t_j for $j \in \{0, 1, 2\}$. The elements $x_\eta = \eta t_j \eta^{-1} {t_j}^{-1}$ corresponding to the following orbit representatives are M-strictly admissible:*

- *Fixing the involution to be θ_{t_0}:* $e; w_\alpha; w_\alpha x_\alpha(1); w_\alpha w_\beta w_\alpha x_\alpha(1); w_0 x_{3\alpha+2\beta}(1)$
- *Fixing the involution to be θ_{t_1}:* $e; w_\alpha$
- *Fixing the involution to be θ_{t_2}:* $e; w_\alpha; w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)$; $w_0 x_{\alpha+\beta}(1); w_0 x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3)$

In particular, the x corresponding to the open orbit with representative $w_0 x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3)$ is M-admissible when choosing the involution to be θ_{t_2}.

Proof. This fact is easily verified with *SageMath* (see the file "admi-conditions-Pb"). We can compare the matrices M and $\theta_x(M)$ for each x, or notice (using *SageMath*) that the x elements listed in the statement of the proposition are all elements of the maximal torus. Indeed, in each case, they correspond either to the identity or to one of the three t_i, $i \in \{0, 1, 2\}$ except for the t_j which characterizes the involution chosen. Then, we trivially obtain $\theta_x(M) = t_k t_j M t_j^{-1} t_k^{-1} = M$. We further notice that, $L = M \cap xMx^{-1}$ (see page 2 of [20]), hence $L = M$. Notice finally that the condition $x \in L.w$ is obviously satisfied for them since they are torus elements. □

Proposition 7.3. *Let us denote $w_1 = w_\beta w_\alpha w_\beta w_\alpha$. Let $M = M_\alpha$, and the involution be given θ_{t_j}, i.e by conjugation by one of the t_j for $j \in \{0,1,2\}$. The elements $x_\eta = \eta t_j \eta^{-1} t_j{}^{-1}$ corresponding to the following orbit representatives are M-strictly admissible:*

- *Fixing the involution to be θ_{t_0}: $e; w_\beta w_\alpha$*
- *Fixing the involution to be θ_{t_1}: $e; w_\beta w_\alpha; w_\beta w_\alpha x_{3\alpha+\beta}(1); w_1 x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3)$*
- *Fixing the involution to be θ_{t_2}: $e; w_\beta w_\alpha; w_\beta w_\alpha x_\alpha(1); w_\beta w_\alpha w_\beta w_\alpha x_\alpha(1)$; $w_\beta w_\alpha w_\beta w_\alpha x_{3\alpha+2\beta}(1)$*

In particular, the x corresponding to the open orbit with representative $w_1 x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3)$ is M-admissible when choosing the involution to be θ_{t_1}.

Proof. This fact is easily verified with *SageMath* (see the file "admi-conditions-Pa"). We can compare the matrices M and $\theta_x(M)$ for each x, or notice (using *SageMath*) that the x elements listed in the statement of the proposition are all elements of the maximal torus. Indeed, in each case, they correspond either to the identity or to one of the three t_i, $i \in \{0,1,2\}$ except for the t_j which characterizes the involution chosen. Then, we trivially obtain $\theta_x(M) = t_k t_j M t_j^{-1} t_k^{-1} = M$. We further notice that, $L = M \cap xMx^{-1}$ (see page 2 of [20]), hence $L = M$. Notice finally that the condition $x \in L.w$ is obviously satisfied for them since they are torus elements. □

Proposition 7.4. *If $P = P_\beta$, the x elements corresponding to the orbits representatives e and w_α are closed for any choice of involution among the θ_{t_j}, $j \in \{0,1,2\}$.*
If $P = P_\alpha$, the x elements corresponding to the orbits representatives e and $w_\beta w_\alpha$ are closed for any choice of involution among the θ_{t_j}, $j \in \{0,1,2\}$.

Proof. As we observed in the proof of Proposition 7.3, the x elements corresponding to the orbits representatives e and w_α (resp. $w_\beta w_\alpha$) are torus elements, it follows trivially that $\theta_x(P) = P$. We further deduce that $U_x = U$. Finally, Lemma 6.3 of [20]) shows that whenever $x \in X \cap N_{G,\theta}(M)$, then $P_x = M_x \rtimes U_x$. Obviously then, the modular character δ_{P_x} is just δ_{P_β} (resp. δ_{P_α}). □

Proposition 7.5. *Let us denote w_0 the element $w_\alpha w_\beta w_\alpha w_\beta w_\alpha$. The stabilizer of the representative $w_0[0,1,r_3,0]$ in SO_4 is isomorphic to one its subgroup SO_2 and is therefore of minimal dimension. Therefore $P_\beta w_0[0,1,r_3,0]\mathrm{SO}_4$ (resp. $P_\alpha w_1[0,1,r_3,0]\mathrm{SO}_4$) is open in $\mathbf{G}_2$, and $\mathcal{O}_{w_0[0,1,r_3,0]}$ (resp. $\mathcal{O}_{w_1[0,1,r_3,0]}$) is an open orbit. The different SO_2 in SO_4 are parametrized by the square classes $F^\times/(F^\times)^2$ and each gives rise to a given open orbit.*

Proof. There exists a torus element $t_e(s, ts^{-1})$ in T_β with $t = s$ which satisfies the following equation: $t_e x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3) t_e^{-1} = x_{\alpha+\beta}(1) x_{3\alpha+\beta}(1)$ and $w_0 t_e x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3) t_e^{-1} = w_0 t_e w_0^{-1} w_0 x_{\alpha+\beta}(1) x_{3\alpha+\beta}(r_3) t_e^{-1}$ with $w_0 t_e w_0^{-1}$

denoted t' (which is some element in the torus only depending on the variable t) of the torus T_β and since we look at $w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)$ in a double coset, multiplying on the left by t_e^{-1} and on the right by t'^{-1} is harmless, so $w_0 t_e x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e^{-1} \sim t' t_e^{-1} w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e t'^{-1}$ but also $\sim w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$.
So we have found an element in the torus which stabilizes the orbit $w_0[0,1,r_3,0]$ and depends on only one variable (i.e is of dimension one). Further, we notice that t_e acts as square $x_{3\alpha+\beta}(r_3)$. The square class $r_3 t^2$ is the quadratic form $e \to r_3 N(e)$, with $e \in E$, with r_3 not a square, attached to E the quadratic extension of F. We let $V = E + (-E)$, where $(-E)$ is the same vector space with the negative quadratic form, be the split ambient non-degenerate 4-dimensional quadratic space and W a two-dimensional quadratic subspace of V such that $\mathrm{SO}(E) \cong \mathrm{SO}(W) \subset \mathrm{SO}(V) \cong \mathrm{SO}_4$. Therefore, they are as many $\mathrm{SO}(W)$ as they are quadratic extensions of F (see also the end of the proof of Lemma 6.1). Each being the stabilizer of minimal dimension (dimension 1) of $w_0[0,1,r_3,0]$, each gives rise to an open orbit. □

8. SO_4-distinguished induced representations of G_2

As we are approaching our final results, one question remains untouched: The question of which characters χ of $\mathrm{SO}_4(F)$ are we using when we apply the Proposition 3.1, and how can they be seen, at first, as characters of the Levi M_β isomorphic to GL_2.

8.1. Characters of SO_4 and characters of GL_2. Let us recall here how $\mathrm{GL}_2 = \mathrm{GL}_2(F)$ sits inside SO_4. $\mathrm{GL}_2 \times \mathrm{GL}_2$ operates on $X = \mathrm{M}_2(F)$ by left and right regular representations, preserving determinant (a quadratic form on the 4-dimensional space X) up to scalars:

$$(g,h)X = gXh^{-1}$$

It is known that the split form of SO_4 is isomorphic to $\mathrm{SL}_{2,s} \times SL_{2,l}/\Delta < \pm 1 >$, hence there are a few options for the characters of GL_2 to be related to those of SO_4.

Lemma 8.1. *We assume the characteristic of F is different from 2. The characters of $\mathrm{SO}_4(F)$ are the characters of $F^\times/F^{\times 2}$, i.e are quadratic characters of $F^\times$.*

Proof. The characters of SO_4 come from the spinor norm as discussed by Serre in [26], 3.2 b). Let q be a nondegenerate quadratic form of rank n. There exists a cohomology exact sequence

$$\mathrm{Spin}_q(F) \to \mathrm{SO}_q(F) \to F^\times/F^{\times 2} \to H^1(F, \mathrm{Spin}_q) \to H^1(F, \mathrm{SO}_q) \to \mathrm{Br}_2(F)$$

We have $H^1(F, \mathrm{Spin}_q) = 0$ by a result of Kneser [15] (see also [26] 3.1) and using the fact that Spin_q is simply connected. Therefore, we have the sequence $\mathrm{Spin}_q(F) \to \mathrm{SO}_q(F) \to F^\times/F^{\times 2} \to 0$. Since $\mathrm{Spin}_q(F)$ is its own commutator, it has no non-trivial characters, and therefore any complex character of $\mathrm{SO}_q(F)$ must be trivial on $\mathrm{Spin}_q(F)$. Finally, by the sequence above, it means any complex character of $\mathrm{SO}_q(F)$ can be identified with a character of $F^\times/F^{\times 2}$ (a set of cardinality 4 by the arguments recalled at the end of Lemma 6.1), i.e with a quadratic character of $F^\times$. $\square$

Theorem 8.2 (Closed orbit)**.** *Let χ be a character of $\mathrm{SO}_4(F)$. It is a quadratic character of $F^\times$. It can be seen as a character of GL_2 (those are given by $\chi \circ \det$ for a quasi-character χ of $F^\times$). Let P_β (resp. P_α) denote the maximal parabolic corresponding to the root β (resp. α). The parabolic induced representations of G_2 which are (SO_4, χ)-distinguished include the following representations:*

- *The induction from P_β to G_2 of the reducible principal series $I(\chi\delta_{P_\beta}^{1/2}|.|^{-1/2} \otimes |.|)$ of GL_2.*
- *The induction from P_α to G_2 of the reducible principal series $I(\chi\delta_{P_\alpha}^{1/2}|.|^{1/2} \otimes \chi\delta_{P_\alpha}^{1/2}|.|^{-1/2})$ of GL_2.*
- *The induced representation $I_{P_\beta}^{G_2}((\chi \circ \det)\delta_{P_\beta}^{1/2})$*
- *The induced representation $I_{P_\alpha}^{G_2}((\chi \circ \det)\delta_{P_\alpha}^{1/2})$.*

Proof. Let us first remark that the description of the characters of SO_4 results from Lemma 8.1. Secondly, we have identified two closed parabolic orbits among the thirteen (resp. twelve) orbits: the one associated to the element $x = e$ (which is, by definition, closed), and another, see Proposition 7.4.

Applying Proposition 3.1 we know that if σ is $(M_x, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$-distinguished then $\mathrm{Ind}_P^G(\sigma)$ is (H, χ)-distinguished. Since $L = M \cap \eta\theta(\eta^{-1}M\eta)\eta^{-1} = M$, $M_x = L_x = M \cong \mathrm{GL}_2$ in this case. Notice, also from Proposition 7.4, that $\delta_{Q_x} = \delta_{P_\beta}$ (resp. $\delta_{Q_x} = \delta_{P_\alpha}$). We are therefore looking at inducing GL_2-representations (denoted σ in the Proposition 3.1) which are $(\mathrm{GL}_2, \delta_{P_\beta}^{1/2}\chi)$-distinguished (resp. $(\mathrm{GL}_2, \delta_{P_\alpha}^{1/2}\chi)$-distinguished). We then use Proposition 4.1 and the following Remark 4.2. $\square$

Theorem 8.3 (Open orbit)**.** *Let P_β (resp. P_α) denote the maximal parabolic corresponding to the root β (resp. α). Let us define the involution θ whose fixed points are SO_4 to be either θ_{t_2} if the inducing parabolic is P_β or θ_{t_1} if the inducing parabolic is P_α. Then the induced representations of G_2 which are SO_4-distinguished include the following representations:*

- *The induction from P_β to G_2 of the reducible principal series $I(|.|^{-1/2} \otimes |.|)$ of GL_2.*
- *The induction from P_α to G_2 of the reducible principal series $I(|.|^{1/2} \otimes |.|^{-1/2})$ of GL_2.*
- *The induced representation $I_{P_\beta}^{G_2}(\mathbf{1})$ for the trivial character $\mathbf{1}$*
- *The induced representation $I_{P_\alpha}^{G_2}(\mathbf{1})$ for the trivial character $\mathbf{1}$.*

Proof. In Propositions 7.2 and 7.3, we noticed that, when taking the involution specified in the statement above, the element x corresponding to $w_0[0,1,r_3,0]$ was M_β-admissible (resp. M_α-admissible). Thus, the conditions to apply Proposition 3.2 are satisfied, and we combine it with Proposition 4.1 and the following Remark 4.2. $\square$

Theorem 8.4 (Distinguished induced parabolic representations and admissible orbits)**.** *We take the involution θ defining $\mathrm{SO}_4(F) = \mathbf{G}_2^\theta(F)$ to be of the form θ_{t_i} for $i \in \{0,1,2\}$, as well as the expressions of the Levi subgroups as defined in the Subsection 7.1. The parabolically induced representations from the parabolic subgroups P_β or P_α of G_2 distinguished by SO_4 whose linear forms arise from admissible, open or closed, orbits are necessarily of the form given in the Theorems 8.2 and 8.3.*

Proof. We apply the Proposition 3.1 and refer to Proposition 7.2 in [20] for the open orbit. Notice that the condition $x \in N_{G,\theta}(M)$ is equivalent to the condition of strict-admissibility as defined in the Definition 3.5.
First, we have shown that following this definition, the only strictly-admissible elements are the one given in the Propositions 7.3 and 7.2. Thus, to apply the Proposition 3.1 and , the following condition is necessarily satisfied: $\theta_x(M) = M$ and $M_x = M \cong GL_2$. In other words, the case where $M_x = T$ does not occur. The case where the orbit is closed, and $M \cong GL_2$ was treated in the Theorem 8.2. $\square$

Appendix A. Conventions for $\mathbf{G}_2$ used for SageMath computations

The appendix contains the necessary background information that allows one to embed the exceptional group $\mathbf{G}_2$ into the general linear group GL(8). Realizing $\mathbf{G}_2$ as the automorphism group of an eight-dimensional Cayley algebra $\mathcal{C}$, and then computing the matrices of elements in root groups with respect to a chosen basis for $\mathcal{C}$ is enough to produce the embedding. Such an embedding was used extensively to carry out the calculations in *SageMath* that are used throughout the present paper. The material in this appendix has been graciously provided by Steven Spallone who in turn would like to acknowledge Gordan Savin for getting him started. Any errors in what follows are the responsibility of the author.

Preliminaries on the Cayley algebra. First, we describe the split Cayley algebra $\mathcal{C}$ over F. Let $\mathrm{M}_2(F)$ be the algebra of 2×2 matrices with entries in F. As an F-vector space, $\mathcal{C} = \mathrm{M}_2(F) \oplus \mathrm{M}_2(F)$ and a typical element of $\mathcal{C}$ can be written as a pair $c = (x \mid y)$, where $x, y \in \mathrm{M}_2(F)$ (see 1.5 in [27], for instance). Multiplication on $\mathcal{C}$ is given by

$$(x \mid y)(x' \mid y') = (xx' + \mathrm{adj}(y')y \mid y'x + y\,\mathrm{adj}(x')),$$

for all $(x \mid y), (x' \mid y') \in \mathcal{C}$. Here $\mathrm{adj}(x)$ is the usual adjugate matrix, which agrees with $(\det x)\cdot x^{-1}$ when $x \in \mathrm{M}_2(F)$ is invertible. The algebra $\mathcal{C}$ has an identity $e = (I_2 \mid 0)$, where I_2 is the 2×2 identity matrix, and the subspace spanned by e is the centre of $\mathcal{C}$.

There is a conjugation map on $\mathcal{C}$ given by

$$\overline{(x \mid y)} = (\mathrm{adj}(x) \mid -y),$$

and a norm map $N : \mathcal{C} \to F$ given by

$$N((x \mid y)) = \det x - \det y.$$

For any $c \in \mathcal{C}$, the trace of c is defined to be $c + \overline{c}$. If $c = (x \mid y)$, then

$$c + \overline{c} = \mathrm{tr}\,(x)e,$$

which we identify with the usual trace $\mathrm{tr}\,(x) \in F$ of x. Thus, we abuse notation and write $\mathrm{tr} : \mathcal{C} \to F$ for the map $c \mapsto c + \overline{c}$. The bilinear form determined by N, namely the pairing defined for $c, d \in \mathcal{C}$ by

$$\langle c, d \rangle = N(c + d) - N(c) - N(d)$$

is non-degenerate. Observe that $\langle c, d \rangle = \mathrm{tr}\,(c\overline{d})$ for all $c, d \in \mathcal{C}$; in particular, $\langle c, e \rangle = \mathrm{tr}\,(c)$, for all $c \in \mathcal{C}$.

The Automorphism Group of $\mathcal{C}$. Let G be the group of automorphisms of the algebra $\mathcal{C}$. It is now well known that G is a split semisimple algebraic group of type $\mathbf{G}_2$ (this was first proved by E. Cartan [4]). By [27], the elements of G stabilizing $\mathcal{A}_2 = \left(\begin{array}{cc|cc} * & * & 0 & 0 \\ * & * & 0 & 0 \end{array}\right)$ are of the form $\varphi_{c,p}$, where

$$\varphi_{c,p}(x \mid y) = (cxc^{-1} \mid pcyc^{-1}),$$

with $c \in \mathrm{GL}_2(F)$ and $p \in \mathrm{SL}_2(F)$.

Let $\lambda_1, \lambda_2 \in F^\times$. Let $a_{\lambda_1,\lambda_2} \in \mathrm{GL}_2(F)$ be the diagonal matrix

$$a_{\lambda_1,\lambda_2} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Then define the element $\gamma(\lambda_1, \lambda_2) \in G$ via

$$\gamma(\lambda_1, \lambda_2)(x \mid y) = \Big(\mathrm{Int}(a_{\lambda_1,\lambda_2})(x) \mid a_{\lambda_2,{\lambda_2}^{-1}}\,\mathrm{Int}(a_{\lambda_1,\lambda_2})(y)\Big), \tag{A.1}$$

for all $(x \mid y) \in \mathcal{C}$. Recall that $\mathrm{Int}(g)(x) = gxg^{-1}$, for any $g \in \mathrm{GL}_2(F)$ and $x \in \mathrm{M}_2(F)$.

Let $T = \{\gamma(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in F^\times\}$; then T is a maximal torus of G. Let $\gamma = \gamma(\lambda_1, \lambda_2) \in T$. Define $\alpha(\gamma) = \lambda_1{\lambda_2}^{-1}$ and $\beta(\gamma) = {\lambda_2}^2{\lambda_1}^{-1}$. Then we have

$$\begin{aligned} (\alpha + \beta)(\gamma) &= \lambda_2, \\ (2\alpha + \beta)(\gamma) &= \lambda_1, \\ (3\alpha + \beta)(\gamma) &= {\lambda_1}^2{\lambda_2}^{-1}, \qquad\qquad \text{and} \\ (3\alpha + 2\beta)(\gamma) &= \lambda_1\lambda_2. \end{aligned}$$

Let $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and define $w_G \in G$ by

$$w_G((x \mid y)) = (sxs^{-1} \mid sys^{-1}).$$

Then conjugation by w_G acts by inversion on T, and thus represents the longest Weyl group element of T in G.

Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G can be identified with the algebra of derivations of $\mathcal{C}$. Recall that a derivation of $\mathcal{C}$ is a linear map $D : \mathcal{C} \to \mathcal{C}$ so that

$$D(cd) = D(c)d + cD(d),$$

for all $c, d \in \mathcal{C}$.

The adjoint action of G on $\mathfrak{g}$ is given by $(\mathrm{Ad}(g)D)(c) = g(D(g^{-1}c))$, for all derivations $D \in \mathfrak{g}$ and $c \in \mathcal{C}$. Let $\mathfrak{t}$ denote the Lie algebra of the torus T. Let $\lambda_1, \lambda_2 \in F$ and let $\gamma(\lambda_1, \lambda_2) \in \mathfrak{t}$. Let

$$t = a_{\lambda_1,\lambda_2} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

It is easy to see that for an element $(x \mid y) \in \mathcal{C}$ we have

$$^{\gamma(\lambda_1,\lambda_2)}(x \mid y) = ([t, x], \mathrm{tr}\,(t)y - yt).$$

One-parameter root subgroups of G_2.

Root subgroup and other objects related to α. For any $t \in F$, define $u_\alpha(t), u_{-\alpha}(t) \in G$ by

$$u_\alpha(t)(x \mid y) = (\mathrm{Int}(V(t))x \mid yV(-t))$$

and

$$u_{-\alpha}(t)(x \mid y) = (\mathrm{Int}(\overline{V}(t))x \mid y\overline{V}(-t)),$$

for all $(x \mid y) \in \mathcal{C}$, where

$$V(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{V}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Explicitly,

$$u_\alpha(t) \left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array} \right) = \left(\begin{array}{cc|cc} x_1 + tx_3 & x_2 + t(x_4 - x_1) - t^2 x_3 & y_1 & y_2 - ty_1 \\ x_3 & x_4 - tx_3 & y_3 & y_4 - ty_3 \end{array} \right)$$

Let n_α be a representative of the reflection in W_{G_2} corresponding to the root α. Following Chevalley's recipe (see [12, §32.3], for instance), one easily computes that $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ is given by

$$n_\alpha(x \mid y) = (sxs^{-1} \mid ys)$$

where $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, as above. Note that $n_\alpha^2 = \gamma(-1, -1)$, and that

$$n_\alpha \gamma(\lambda_1, \lambda_2) n_\alpha^{-1} = \gamma(\lambda_2, \lambda_1).$$

A.0.1. *Root subgroup and other objects related to* β. If $a, b \in \mathcal{C}$, define the map $L_{a,b} : \mathcal{C} \to \mathcal{C}$ by

$$L_{a,b}(c) = \langle c, a\rangle b - \langle c, b\rangle a,$$

for all $c \in \mathcal{C}$. Take

$$x_0 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), w_0 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and define $D_\beta = L_{w_0,x_0}$. Put

$$x_0' = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), w_0' = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

and define $D_{-\beta} = L_{w_0', x_0'}$. It is straightforward to check that D_β and D_β are the root vectors for the roots β and $-\beta$ in the Lie algebra $\mathfrak{g}$ of G.

Then for $t \in F$ and $c \in \mathcal{C}$, define

$$u_\beta(t)(c) = c + L_{w_0,x_0}(tc).$$

and

$$u_{-\beta}(t)(c) = c + L_{w_0',x_0'}(tc).$$

Explicitly,

$$u_\beta(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2 & y_1 + tx_2 & y_2 \\ x_3 - ty_4 & x_4 & y_3 & y_4 \end{array}\right)$$

and

$$u_{-\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2 + ty_1 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 - tx_3 \end{array}\right).$$

Let n_β denote the representative of the reflection corresponding to β in W_{G_2}. Following Chevalley's recipe, as above, if we set $n_\beta = u_\beta(1)u_{-\beta}(-1)u_\beta(1)$, then

$$n_\beta(c) = c + \langle c, w - x'\rangle x + \langle c, w' - x\rangle w - \langle c, w' + x\rangle x' + \langle c, w + x'\rangle w'.$$

Explicitly

$$n_\beta\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & -y_1 & x_2 & y_2 \\ -y_4 & x_4 & y_3 & x_3 \end{array}\right).$$

Note that $n_\beta^2 = \gamma(1, -1)$, and that

$$n_\beta \gamma(\lambda_1, \lambda_2) n_\beta^{-1} = \gamma(\lambda_1, \lambda_1\lambda_2^{-1}).$$

A.0.2. *More root subgroups.* The formula $\mathrm{Int}(n_\alpha)u_{3\alpha+\beta}(t) = u_\beta(t)$ gives

$$u_{3\alpha+\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2 - ty_3 & y_1 & y_2 - tx_3 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right).$$

The formula $\mathrm{Int}(n_\beta)u_{\alpha+\beta}(t) = u_\alpha(-t)$ gives

$$u_{\alpha+\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 + ty_4 & x_2 & y_1 + t(x_4 - x_1) - t^2 y_4 & y_2 + tx_2 \\ x_3 + ty_3 & x_4 - ty_4 & y_3 & y_4 \end{array}\right).$$

The formula $\mathrm{Int}(n_\alpha)u_{2\alpha+\beta}(t) = u_{\alpha+\beta}(-t)$ (see for instance [12][3.35] gives

$$u_{2\alpha+\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 - ty_3 & x_2 + ty_4 & y_1 - tx_3 & y_2 + t(x_4 - x_1) + t^2 y_3 \\ x_3 & x_4 + ty_3 & y_3 & y_4 \end{array}\right).$$

The formula $\mathrm{Int}(n_\beta)u_{3\alpha+2\beta}(t) = u_{3\alpha+\beta}(-t)$ gives

$$u_{3\alpha+2\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2 & y_1 - ty_3 & y_2 - ty_4 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right).$$

A.1. **Embedding $\mathbf{G}_2$ into** $\mathrm{GL}(8)$**.** The algebra $\mathcal{C}$ is eight-dimensional and has ordered basis $\mathcal{B} = \{e_{11}, e_{21}, e_{31}, e_{41}, e_{12}, e_{22}, e_{32}, e_{42}\}$ where

$$e_{11} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad e_{12} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$e_{21} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad e_{22} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$e_{31} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \qquad e_{32} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$e_{41} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad e_{42} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Calculating the matrices of $\gamma(\lambda_1, \lambda_2) \in T$, and the root groups described above with respect to the basis $\mathcal{B}$ is enough to embed $G \cong \mathbf{G}_2$ into $\mathrm{GL}(8)$.

Appendix B. The code

The code is organized in two branches: one "main" branch where all files needed to justify the results presented in this article are available; the second branch, "old strategy" contains the same computations as in the other branch but using the matching elements rather than the x-elements.

One subsidiary result we verified with Sage was the matching of each orbit representative η, to a unique element in the double cosets space $W_\beta \backslash W / W_\beta = \{e, w_\alpha, w_\alpha.w_\beta.w_\alpha, w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha\}$, where W_β denotes $W_{M_\beta} =< w_\beta >$.

B.1. **The matching.** Following Lemma 3.1 in [20], we know each orbit representative η, as given in the previous section, corresponds to a unique element in the double cosets space

Recall the following map from Subsection 3.1:

$$\iota_M : P\backslash X \to {}_M W_{M'} \tau^{-1} \cap \mathcal{S}_0(\theta),$$

where $\mathcal{S}_0(\theta) = \{w \in W : w\theta(w) = e\}$ is the set of twisted involutions in the Weyl group, and the definition of τ was recalled in the above subsection. For an element $n \in \tau$, $\theta'(g) = n^{-1}\theta(g)n$. Here M' is the $\theta' = \theta$ conjugate of M. In our context, first the set of twisted involutions is just the set of involutions, as our involution consists in the conjugation by an order two element of the torus, secondly out of the twelve elements in W, seven are indeed involutions. This is easily verified with *SageMath*, although our readers need to pay attention that the product ww^{-1} might not necessarily be the identity matrix, but can also be an order two element of the torus.

Fix $x \in X$, and recall $x = \eta.e = \eta e\theta(\eta)^{-1}$. η, x, match some unique elements in the double cosets $W_\beta\backslash W/W_\beta$: $w = \iota_M(P\dot{x})$. This uniqueness follows from a statement at the bottom of p216 in [20] and Proposition 7.1. Offen uses expressions which depend on P' and M' but since θ_{t_1} stabilizes M, $M' = M$ as we have verified this matching using θ_{t_1}.

Our first step while dealing with this project was to verify this matching and to do so we have used the involution given by θ_{t_1} (this verification was not done for θ_{t_0} or θ_{t_2}). Concretely, we are verifying in *SageMath* (again the code is available in the github file for the convenience of the reader) the following equations.

- $PxP = PwP$
- $t_1 = w * t_1 * w$
- w is left and right $W_{(M_\beta)}$-reduced.

Look at the first point above in *SageMath*: we compare each left side of the equation to the four elements in $W_\beta\backslash W/W_\beta$ and eliminate progressively variables to reach some contradiction for all elements but one which is the match. The results are given in the bracket below:

(B.1)

$$\begin{cases} \eta & x=\eta.\theta(\eta)^{-1} & W_\beta\backslash W/W_\beta \\ e & e & e \\ w_\alpha & w_\alpha.w_\alpha & e \\ w_\alpha x_\alpha(1) & w_\alpha.x_\alpha(2).w_\alpha & w_\alpha \\ w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1) & w_\alpha * w_\beta * w_\alpha * x_{3\alpha+\beta} * x_{3\alpha+\beta} * w_\alpha * t_1 * w_\beta^{-1} * t_1 * w_\alpha & w_\alpha.w_\beta.w_\alpha \\ w_\alpha w_\beta w_\alpha x_\alpha(1) & w_\alpha * w_\beta * w_\alpha * x_\alpha * x_\alpha * w_\alpha * t_1 * w_\beta^{-1} * t_1 * w_\alpha & w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha \\ w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1) & & w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha \\ w_0 x_{\alpha+\beta}(1) & & w_\alpha \\ w_0 x_{3\alpha+2\beta}(1) & & w_\alpha.w_\beta.w_\alpha \\ w_0 x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1) & & w_\alpha.w_\beta.w_\alpha \\ w_0[0,1,r_3,0] & & w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha \\ w_0 & w_0 t_1 w_0^{-1} t_1 & w_0 \end{cases}$$

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Mathematical Sciences 566, University of Calgary, Calgary, Alberta T2N 1N4, Canada

Email address: sarah.dijols@hotmail.fr